2022 IEEE 61st Conference on Decision and Control (CDC)
December 6-9, 2022. Cancún, Mexico

Constrained trajectory synthesis via quasi-interpolation

Siddhartha Ganguly[1], Nakul Randad[2], Debasish Chatterjee[1], Ravi Banavar[1]

2022 IEEE 61st Conference on Decision and Control (CDC) | 978-1-6654-6761-2/22/$31.00 ©2022 IEEE | DOI: 10.1109/CDC51059.2022.9992892

***Abstract*— In this article we introduce QuITO — Quasi-Interpolation based Trajectory Optimization, a direct multiple shooting algorithm to solve a class of constrained nonlinear minimum energy optimal control problems. This technique is based on the theory of *approximate approximations* – a quasi-interpolation scheme. We parameterize the control trajectory using the quasi-interpolation formula, and we discretize the optimal control problem using the collocation points on a uniform cardinal grid, thereby transcribing the optimal control problem (OCP) into a nonlinear program (NLP). Several examples are provided to show the numerical fidelity of the algorithm.**

***Index Terms*— optimal control, quasi-interpolation, direct multile shooting, collocation**

I. Introduction

Optimal control theory provides engineers with a powerful tool for synthesising control in a constrained environment, with the option to impose constraints at the synthesis stage while optimizing an appropriate cost function. One of the challenging tasks in this context is to design efficient numerical algorithms to solve constrained optimal control problems. Typically there are two school of thoughts:

- *The Pontryagin Maximum Principle* (PMP) [14] gives first-order necessary optimality conditions for optimal control problems and induces the so-called numerical *indirect method* which consists of determining the optimal control trajectory by numerically solving a two-point-boundary-value-problem (TPBVP), see [2].
- *Direct methods* view the optimal control problem through the glasses of approximation and optimization and discretizes the control problem itself, transcribing it into a nonlinear program. This whole class of trajectory synthesis methods goes by the name of *collocation* [13].

Using the first approach, i.e., the classical PMP, it is often difficult to deal with a general class of problems involving, for example, state constraints or mixed state-control constraints. On the other hand, most of these difficulties gets eliminated when we deal with OCPs via the direct method while compromising accuracy in certain cases [24]. The feature here is the *underlying approximation scheme*, which reduces the burden of searching admissible trajectories in some function space to some finite dimensional vector space, via a nonlinear program. It is evident that the ensuing error estimate that the approximation scheme can provide is of utmost importance.

Perhaps one of the most well known class of direct methods for solving constrained optimal control problems are the *pseudospectral* or the *orthogonal collocation* methods [22]. Broadly, there are three types of collocation points namely – Legendre-Gauss (LG), Legendre-Gauss-Radau (LGR) and Legendre-Gauss-Lobato (LGL), one can choose to discretize the problem [8], with the employment of a viable approximation scheme at the level of only control or both control and state variables. Pseudospectral collocation methods employ global/local orthogonal polynomials to approximate the states/control. The system dynamics and the constraints are enforced at the collocation points. These algorithms essentially rely on Hilbert space methods [6], and the convergence results of the interpolation techniques are $\mathbb{L}^2$-based. There are a few recent results [10], [11] that provide local $\mathbb{L}^\infty$ convergence results for a class of optimal control problems.

While the $\mathbb{L}^2$-based interpolation methods provides accurate approximations, they suffer from certain defects, for example Gibbs's phenomenon. We are interested in an alternative $\mathbb{L}^\infty$ approximation procedure. To this end, we establish a direct multiple shooting method with control parameterization via *approximate approximations* (see §III-A for details, a book-length treatment can be found in [19]), which provides approximation guarantees in the $\mathbb{L}^\infty$-norm.

Our contributions

The primary contributions of this article are:

1) We establish a direct multiple shooting method (we will call it QuITO — Quasi-Interpolation based Trajectory Optimization) leveraging control parameterization by means of a quasi-interpolation scheme,
2) We record several benchmark numerical examples to show the effectiveness of the QuITO algorithm.

Our purpose herein is to serve a source of preliminary ideas which will be further developed in our subsequent works.

II. Problem Statement

Let us consider a nonlinear time-varying controlled system on a fixed time interval $[t_0, t_{\mathrm{f}}]$, $t_0 < t_{\mathrm{f}}$, modeled by the ordinary differential equation

$$\dot{x}(t) = f\big(x(t)\big) + G\big(x(t)\big)u(t) \text{ for a.e. } t \in [t_0, t_{\mathrm{f}}], \quad \text{(II.1)}$$

with the following data:

((II.1)-a) $x(t) \in \mathbb{R}^d$ is the vector of states and

((II.1)-b) $u(t) \in \mathbb{R}^m$ is the control action at time t.

*Siddhartha Ganguly is supported by the PMRF grant RSPMRF0262, from the Ministry of Human Resource Development, Govt. of India.

[1]Siddhartha Ganguly, Debasish Chatterjee and Ravi Banavar are with Systems & Control Engineering, IIT Bombay, Powai, Mumbai 400076, India, {sganguly,dchatter,banavar}@iitb.ac.in

[2]Nakul Randad is with the Department of Aerospace Engineering, IIT Bombay, Powai, Mumbai 400076, India, nakulrandad@iitb.ac.in

Assume that the pair of boundary values $\big(x(t_0), x(t_f)\big)$ lie in a given closed subset of $\mathbb{S} \subset \mathbb{R}^d \times \mathbb{R}^d$, i.e.,

$$\big(x(t_0), x(t_f)\big) \in \mathbb{S} \subset \mathbb{R}^d \times \mathbb{R}^d, \tag{II.2}$$

and that the control input u is constrained to take values in a given nonempty compact, convex subset $\mathbb{U} \subset \mathbb{R}^m$, i.e.,

$$u \in \mathcal{U} := \{u \in \mathbb{L}^\infty \mid u(t) \in \mathbb{U} \text{ for a.e } t \in [t_0, t_f]\} \tag{II.3}$$

A control u is *feasible* if it is measurable,[1] satisfies the control constraint (II.3), and the corresponding solution $x(\cdot)$ of (II.1) satisfies (II.2). Over these feasible controls, we consider the following minimization problem:

$$\begin{aligned} &\underset{u}{\text{minimize}} \quad \mathbb{J}(x,u) := \int_{t_0}^{t_f} \|u(t)\|^2 \, \mathrm{d}t \\ &\text{subject to} \begin{cases} \text{dynamics (II.1)}, \\ h_j\big(x(t)\big) \le 0 \text{ for all } t \in [t_0, t_f],\ j = 1, \ldots, r_0, \\ u \in \mathcal{U}, \\ \big(x(t_0), x(t_f)\big) \in \mathbb{S} \subset \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \end{aligned} \tag{OCP}$$

with the following data:

((OCP)-a) the maps $\xi \mapsto f\big(\xi\big) \in \mathbb{R}^d$, and $\xi \mapsto G\big(\xi\big) \in \mathbb{R}^{d \times m}$ are locally Lipschitz continuous.

((OCP)-b) For $j = 1, \ldots, r_0$ the function $\xi \mapsto h_j\big(\xi\big) \in \mathbb{R}$ is continuously differentiable with locally Lipschitz continuous gradient and the map $u \mapsto h_j \circ x(\cdot)$ is convex for every $j = 1, \ldots, r_0$.

The map $[t_0, t_f] \ni t \mapsto \big(x(t), u(t)\big) \in \mathbb{R}^d \times \mathbb{U}$ is an *admissible state-action trajectory* if it satisfies all the constraints of the (OCP), and $\mathbb{J}(x, u)$ is finite. Moreover, an admissible state-action trajectory (x^*, u^*) is a local minimizer of (OCP) provided that for some $\epsilon > 0$ and for every admissible state-action trajectory (x, u) satisfying $\|x - x^*\|_\infty \le \epsilon$, we have $\mathbb{J}(x^*, u^*) \le \mathbb{J}(x, u)$.

Remark (II.4): (On the existence and regularity of (OCP)) Optimizer regularity is important for various reasons, one of which is the choice of a particular approximation or discretization scheme. Prior knowledge of the function space where the optimizer belongs can greatly influence this choice; for example one may employ an approximation scheme where higher order error estimates and convergence rates can be obtained. The rigorous study of optimizer regularity has attracted a lot of attention since the work [9]. After this, a sequence of optimizer regularity results were reported in [4], [5], [23], for various state and control constrained optimal processes. In particular, we lift a result from [7] which asserts Lipschitz continuity of the optimal control under hypotheses ((OCP)-a)-((OCP)-b) and an additional hypothesis (see Section 3, data (H1)-(H4) in [7]).

Theorem (II.5) ([7, Theorem 3.1]): Let (x^*, u^*) be a normal extremal (see [7, Definition 2.1]) for the problem (OCP). Assume (H1)-(H4), then u^* is Lipschitz continuous.

[1]For us the word 'measurability' always refers to Lebesgue measurability, and 'a.e.' will refer to almost everywhere relative to the Lebesgue measure.

III. MAIN RESULTS

The main focus of this section is the the control parameterized direct multiple shooting algorithm QuITO, which is our primary result. Before doing so, we introduce briefly the theory of approximate approximations.

A. Approximate approximations

Approximation of multivariable functions via quasi-interpolation with data sites $(hj)_{j\in\mathbb{Z}}$ on uniform or square cardinal grid has been studied extensively in the past. Approximate approximations as an approximate quasi-interpolation scheme was introduced in the early 1990s in [16]. The special feature of this procedure is that it is quite accurate without being convergent in a rigorous sense, i.e., the approximation error does not converge to zero as the mesh size say h tends to zero.

We now provide relevant results on approximate approximations pertinent to our work. For simplicity, we restrict ourselves in the realm of data distributions with regular centers [17]. Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be the object of approximation, i.e., the approximand. The idea behind the quasi-interpolation scheme is to represent the approximand u as the sum of scaled and shifted basis functions. For a fixed step size $h > 0$, given a shape parameter $\mathcal{D} \in]0, +\infty[$, and every $x \in \mathbb{R}$ the approximants have the general form

$$\mathcal{M}_{h,\mathcal{D}} u(x) := \frac{1}{\mathcal{D}^{1/2}} \sum_{m \in \mathbb{Z}} u(x_m) \psi\left(\frac{x - x_m}{h\sqrt{\mathcal{D}}}\right), \tag{III.6}$$

where the data points/sites $x_m := hm$ are specified on a cardinal square grid. The generating function $\psi(\cdot)$ belongs to a certain class of *nice* functions with sufficient smoothness attributes. The parameter $\mathcal{D}$ influences the width and thereby the decay rate of the generating functions (consider the Gaussian functions, where $\mathcal{D}$ is the variance). The linear combination of dilated shifts of ψ forms an *approximate partition of unity* i.e.

$$\mathcal{D}^{-1/2} \sum_{m \in \mathbb{Z}} \psi\left((\xi - m)/\sqrt{\mathcal{D}}\right) \approx 1.$$

Notice that this is a point of departure from the plain quasi-interpolants, where the generating function forms an *exact* partition of unity.

Theorem (III.7): [17] Consider a function $u \in \mathcal{C}^{r+1}(\mathbb{R})$, the set of $r+1$ times continuously differentiable functions and data sites $\{x_m : m \in \mathbb{Z}\} \subset \mathbb{R}$. Let the continuous generating function ψ satisfies the *moment condition of order* r*:* $\int_{\mathbb{R}} \psi(y)\,\mathrm{d}y = 1$, $\int_{\mathbb{R}} y^\alpha \psi(y)\,\mathrm{d}y = 0$, for all α, $1 \le [\alpha] < r$; where α is a multi-index, $[\alpha]$ is the length of the multi-index, along with the *decay condition:* $|\psi(x)| \le c_K\big(1 + |x|\big)^{-K/2}$, $x \in \mathbb{R}$, where c_K is some constant, $K > r + 2$. Then

$$\|u - \mathcal{M}_{h,\mathcal{D}} u\|_\infty = \mathcal{O}\big(h^{r+1}\big) + \Delta_0(\psi, h, \mathcal{D}), \tag{III.8}$$

where the term $\Delta_0(\psi, h, \mathcal{D})$ is called the *saturation error*.

Remark (III.9): We can also derive an $\mathbb{L}^\infty$ estimate in a similar fashion when the function u belongs to class of Lipschitz continuous functions. This estimate is useful from the convergence viewpoint as we are working with Lipschitz continuous optimal controls (Theorem (II.5)), see [19, Chapter 2] for more details.

We now summarize the key features of approximate approximations:

- On the right hand side of the $\mathbb{L}^\infty$ *error estimate* (III.8), the first term converges to zero as h goes to zero. The second term $\Delta_0(\psi, h, \mathcal{D})$ is called the *saturation error*, which can be reduced to an arbitrary small number by controlling the shape parameter $\mathcal{D}$.
- For any u in an appropriate class of functions $\mathcal{M}_{h,\mathcal{D}}u$ approximates u arbitrarily well upto some manageable saturation error. This is known as *pseudo convergence*.
- An upper bound on h and a lower bound on $\mathcal{D}$ can be obtained (in some cases precise formulas can be given, see [18]), see [19, Chapter 2] and [19, Chapter 3, Table 1] for precise error estimates, and relevant data.
- The approximant $\mathcal{M}_{h,\mathcal{D}}u$ of u in (III.6) employs an infinite sum to perform the approximation. However, in applications, the summation can be truncated, since the generating functions are chosen to decay fast, see [19, Chapter 2, Section 2.3.2] for more details.

Our approximation engine will be the following one-dimensional summation:

$$\mathcal{M}_{h,\mathcal{D}}u(x) \coloneqq \frac{1}{\sqrt{\mathcal{D}}} \sum_{|m|<+\infty} u(mh)\psi\left(\frac{x-mh}{h\sqrt{\mathcal{D}}}\right), \tag{III.10}$$

where the generating function ψ needs to satisfy the moment and the decay conditions. One of the most common choice is the Gaussian kernel $\psi_G(x) \coloneqq e^{-x^2/\mathcal{D}}/\sqrt{\pi\mathcal{D}}$. Finally, we remark that there are several other choices (see §IV-B) we have at our disposal for the generating function, which can provide higher order error estimates; we refer readers to [18, Lemma 2.6].

B. QuITO: The direct multiple shooting method

We now provide the main result of this article – a direct multiple shooting method to solve the constrained optimal control problem (OCP) numerically. Fix a small number $h > 0$, and choose as basis functions the elements of the set

$$\mathbb{X}_h \coloneqq \left\{ e^{-(x-mh)^2/\mathcal{D}h^2} \,\middle|\, hm \in \Omega, m \in \mathbb{Z} \right\},$$

where Ω is an open domain containing $[t_0, t_{\mathrm{f}}]$ the time horizon. The approximating functions are linear combination of scaled and shifted Gaussian kernels ψ_G centered at the grid points $\{mh \in \Omega\}$. Note that the *discretization* points are $\big(m_i h\big)_{i\in\mathbb{F}}$, where $\mathbb{F} \subset \Omega$ is a finite set. These discretization points are uniformly spaced throughout the grid. We approximate the control trajectory $t \mapsto u(t)$ using the quasi-interpolant (III.10) in the form of an approximant u_A of the control u defined by:

$$u_A(t) \approx \mathcal{M}_{h,\mathcal{D}}u(t) \coloneqq \frac{1}{\sqrt{\mathcal{D}}} \sum_{i\in\mathbb{F}} u\big(m_i h\big)\psi\left(\frac{t-m_i h}{h\sqrt{\mathcal{D}}}\right).$$

Note that $\big(u(m_i h)\big)_{i\in\mathbb{F}}$ are the unknown coefficients corresponding to the the control approximation. We solve the ODE (II.1) in each time interval for $t \in [m_i h, m_{i+1}h]$, starting with a guess initial value η_i

$$\begin{aligned} \dot{x}_i\big(t, \eta_i, u(m_i h)\big) &= f\big(x_i(t, \eta_i, u(m_i h))\big) \\ &\quad + G\big(x_i(t, \eta_i, u(m_i h))\big)u(m_i h) \\ x_i\big(m_i h, \eta_i, u(m_i h)\big) &= \eta_i \quad \text{for all } t \in [m_i h, m_{i+1}h],\ i \in \mathbb{F}. \end{aligned}$$

The problem of joining trajectories together at each η_i which is ensured by the *continuity/defect* condition

$$\eta_{i+1} = x_i\big(m_{i+1}h, \eta_i, u(m_i h)\big), \tag{III.11}$$

is left to the off-the-shelf nonlinear programming (NLP) solver. Thus the continuous time optimal control problem (OCP) can be transcribed into the following NLP

$$\begin{aligned} &\underset{\left(u(m_i h)\right)_{i\in\mathbb{F}}}{\text{minimize}} && \mathbb{J}^{\mathrm{d}} \\ &\text{subject to} && \begin{cases} x_0 = \eta_0, \\ \text{condition (III.11)}, \\ h_j\big(m_i h, \eta_i\big) \le 0,\ i \in \mathbb{F}, \\ u(m_i h) \in \mathbb{U}, \\ \big(\eta_0, \eta_N\big) \in \mathbb{S} \subset \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \end{aligned} \tag{NLP1}$$

where $\mathbb{J}^{\mathrm{d}}$ is the discrete quadrature approximation of the cost in (OCP) and N is the final point of the finite set $\mathbb{F}$.

IV. Numerical Experiments

This section contains a library of numerical experiments showcasing the effectiveness of the QuITO scheme.

A. Bryson-Denham Problem

Consider the well-studied benchmark Bryson-Denham system [3]:

$$\begin{aligned} &\underset{u}{\text{minimize}} && \int_0^1 u(t)^2 \,\mathrm{d}t \\ &\text{subject to} && \begin{cases} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \\ x_1(0) = 0 \quad \text{and } x_2(0) = 1, \\ x_1(1) = 0 \quad \text{and } x_2(1) = -1, \\ x_1(t) \le \frac{1}{9} \quad \text{for all } t \in [0,1], \end{cases} \end{aligned} \tag{IV.12}$$

the analytical solution of the problem (IV.12) can be found in [3]. The problem is challenging from a numerical viewpoint because of the presence of the path constraint $x(t) \le 1/9$. For the numerical simulations we parameterize the control using the expression (III.10) with the basis function ψ_G, and we fix the step size $h = 0.02$ and, $\mathcal{D} = 5$. The numerical solution obtained by using the QuITO algorithm is given in Fig.1 along with the analytic solution for reference. The closeness of the QuITO solution with the analytical solution is noteworthy.

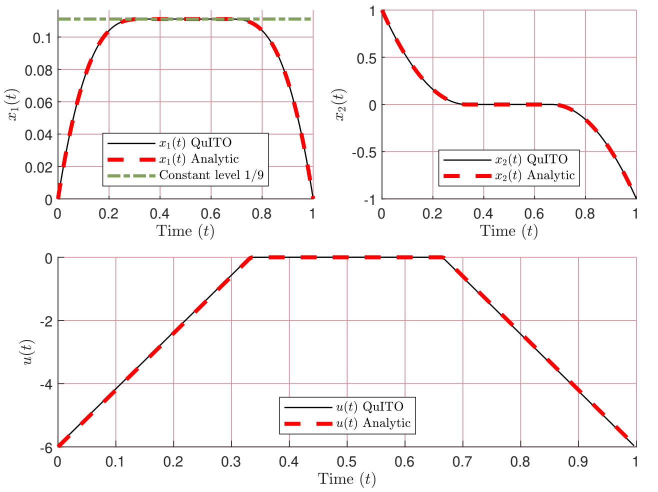

Fig. 1: Time evolution of state and action trajectories with active path constraint on $t \mapsto x_1(t)$.

B. Robot Motion Planning

Consider the problem of planning an optimal path for a robot [20] in an environment with obstacles, given by

$$\begin{aligned}
&\underset{u}{\text{minimize}} \quad \int_0^{t_f} \left(u_1(t)^2 + u_2(t)^2\right) \mathrm{d}t \\
&\text{subject to} \quad \begin{cases}
\dot{x}_1(t) = x_3(t),\ \ \dot{x}_2(t) = x_4(t), \\
\dot{x}_3(t) = u_1(t),\ \ \dot{x}_4(t) = u_2(t), \\
\left(x(0), x(t_f)\right) \in \mathbb{S}, \\
0 \le x_1(t) \le 80,\ \ 0 \le x_2(t) \le 80, \\
|x_3(t)| \le 10,\ \ |x_4(t)| \le 10, \\
80^2 \ge \left(x_1(t) - 40\right)^2 + \left(x_2(t) - 20\right)^2 \ge 10^2, \\
80^2 \ge \left(x_1(t) - 45\right)^2 + \left(x_2(t) - 65\right)^2 \ge 10^2,
\end{cases}
\end{aligned}$$

with $\mathbb{S} := (40\ 5)^\top \times (55\ 70)^\top$ and $t_f = 10$. Fig.2 shows forbidden circular discs that the robot must avoid during its motion. For the numerical simulations we parameterize the control using the expression (III.10) with the following higher order Laguerre polynomial based basis function

$$t \mapsto \eta_6(t) := \frac{1}{\sqrt{\pi}} \left(\frac{15}{8} - \frac{5}{2}|t|^2 + \frac{1}{2}|t|^4\right) e^{-|t|^2}, \quad \text{(IV.13)}$$

which gives an approximation of order $\mathcal{O}(h^6)$ [18]. The associated quasi-interpolant is given by

$$\mathcal{M}^6_{h,\mathcal{D}} u(t) := \frac{1}{\pi\sqrt{\mathcal{D}}} \sum_{i \in \mathbb{F}} u(m_i h) \cdot \left(\frac{15}{8} - \frac{5}{2}\frac{|t - m_i h|^2}{h^2 \mathcal{D}} + \frac{1}{2}\frac{|t - m_i h|^4}{h^4 \mathcal{D}^2}\right) e^{-\frac{|t - m_i h|^2}{\mathcal{D} h^2}}. \quad \text{(IV.14)}$$

We choose $h = 0.2$, $\mathcal{D} = \mathcal{D}(h) := 2M|\log h|/\pi^2$ (see [18]) depending on h and M which is fixed to be 3. The numerical solution obtained by employing the QuITO algorithm is given in Fig.2. The challenging aspect of this problem stems from the presence of multiple path constraints that are active over the course of the robots trajectory.

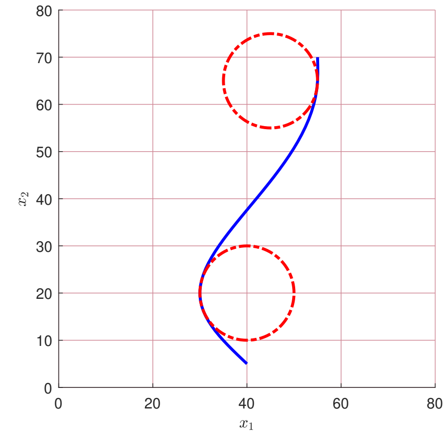

Fig. 2: Optimal trajectory of the robot for the obstacle avoidance maneuver with active circular path constraints.

C. Rayleigh problem

Although the setting of this article is for minimum energy problems we provide some numerical results involving linear quadratic costs in control *and* state variables, to show that a result more general may be possible and will be reported in subsequent articles. We consider the Rayleigh problem [1] with (RP-a) *control* and (RP-b) *mixed state-control* constraints. This problem is known to be numerically stiff [15] and the solution exhibits a limit cycle with the perturbation parameter $p = 0.14$. Another layer of difficulty comes from the mixed state-control constraint, which is known to be burdensome [12] to deal with, specially from trajectory optimization viewpoint. Consider the following optimal control problem

$$\begin{aligned}
&\underset{u}{\text{minimize}} \quad \int_0^{t_f} \left(u(t)^2 + x_1(t)^2\right) \mathrm{d}t \\
&\text{subject to} \quad \begin{cases}
\dot{x}_1(t) = x_2(t), \\
\dot{x}_2(t) = -x_1(t) + x_2(t)\left(1.4 - p x_2(t)^2\right) \\
\qquad\qquad + 4u(t), \\
\left(x_1(0)\ x_2(0)\right)^\top = (-5\ \ -5)^\top, \\
\left(x_1(t_f)\ x_2(t_f)\right)^\top = (0\ 0)^\top, \\
\begin{cases} \text{(RP-a)}\ |u(t)| \le 1, \\ \text{(RP-b)}\ u(t) + \frac{x_1(t)}{6} \le 0. \end{cases}
\end{cases}
\end{aligned}$$

Fix $t_f = 4.5$ and $p = 0.14$. We parameterize the control using the expression (III.10) with the basis function η_6 in §IV-B, and we fix the step size $h = 0.05$ and $\mathcal{D} = 5$. The numerical solution obtained using QuITO with only control constraint (RP-a) is given in Fig.3 and the results with only mixed state-control constraints (RP-b) is given in Fig.4 . We compare our results with the ICLOCS2 solver [21] with LGR pseudospectral collocation and no mesh refinements. See [1], [15] for a clinical comparison with the QuITO solution.

D. Van der Pol Oscillator

We consider the following state constrained optimal control problem which is known to be difficult numerically

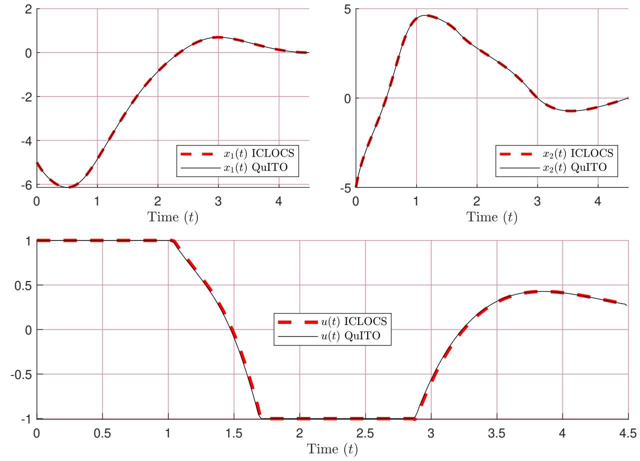

Fig. 3: Time evolution of state and action trajectories with active control constraints (RP-a).

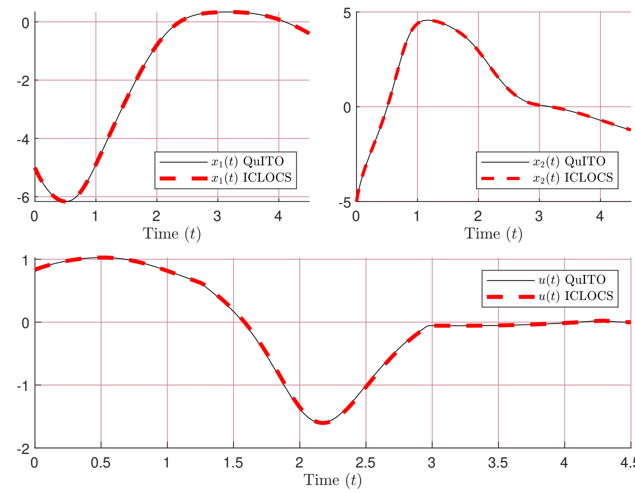

Fig. 4: Time evolution of state and action trajectories with active mixed constraints (RP-b).

because of the nonlinearity of the Van der Pol dynamics, possibility of singular solutions [15], and the presence of state constraints. Consider the following OCP [1]:

$$\begin{aligned} &\underset{u}{\text{minimize}} && \int_0^{t_\mathrm{f}} \big(\langle x(t),\, Qx(t)\rangle + \langle u(t),\, Ru(t)\rangle\big)\,\mathrm{d}t \\ &\text{subject to} && \begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \big(1 - x_1^2(t)\big)x_2(t) - x_1(t) + u(t), \\ \big(x_1(0)\ x_2(0)\big)^\top = (1\ 0)^\top,\ -x_2(t) + p \le 0, \end{cases} \end{aligned}$$

where Q is a 2×2-identity matrix, and the control weight is chosen to be $R = 1$, and fix $t_\mathrm{f} = 5$. We parameterize the control using the expression (III.10) with the basis function ψ_G and we choose $p = -0.4$, $h = 0.05$, and $\mathcal{D} = 5$. The numerical solution obtained by employing the QuITO scheme is given below in Fig.5.

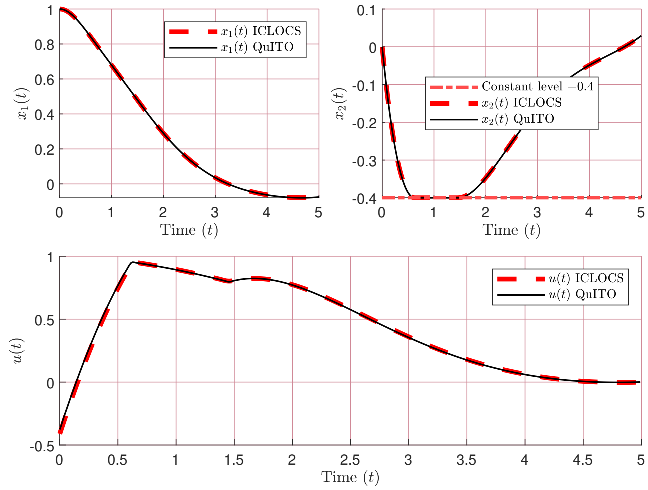

Fig. 5: Time evolution of state and action trajectories with active state constraints

V. Conclusion and Discussion

This article introduced QuITO – a direct trajectory optimization algorithm based on approximate approximation to address certain class of optimal control problems. We provide a library of numerical examples to show the proficiency of the proposed algorithm. Our aim here is not to deal with very general classes of problems, but to report our preliminary investigations, through showcasing the efficiency of the proposed algorithm via several linear and nonlinear moderately high dimensional constrained OCPs. We do not tackle with nonsmooth OCPs in this article, e.g, minimum time, and fuel problems; these cases needs specific detailed attention from the numerical side – for example mesh refinement; our findings on these topics will be reported elsewhere. Our future work involves proving a convergence result, exploring the aforementioned directions and develop a commercial trajectory optimization toolbox based on the QuITO algorithm.

References

[1] J.T. Betts. *Practical Methods for Optimal Control and Estimation Using Nonlinear Programming*, volume 1 of *Advances in Design and Control*. SIAM, 2010.

[2] J.F. Bonnans. The shooting approach to optimal control problems. *11th IFAC Workshop on Adaptation and Learning in Control and Signal Processing*, 46(11):281–292, 2013.

[3] A.E. Bryson and H.C. Ho. *Applied Optimal Control*. John Wiley and Sons., 1975.

[4] A.L. Dontchev and W.W. Hager. Lipschitzian stability in nonlinear control and optimization. *SIAM Journal on Control and Optimization*, 31(3):569–603, 1993.

[5] A.L. Dontchev and W.W. Hager. Lipschitzian stability for state constrained nonlinear optimal control. *SIAM Journal on Control and Optimization*, 36(2):698–718, 1998.

[6] F. Fahroo and I.M. Ross. Advances in Pseudospectral Methods for Optimal Control. In *AIAA guidance, navigation and control conference and exhibit*, page 7309, 2008.

[7] G.N. Galbraith and R.B. Vinter. Lipschitz continuity of optimal controls for state constrained problems. *SIAM journal on control and optimization*, 42(5):1727–1744, 2003.

[8] D. Garg, M. Patterson, W.W. Hager, A.V. Rao, D.A. Benson, and G.T. Huntington. A Unified Framework for the Numerical Solution of Optimal Control Problems using Pseudospectral Methods. *Automatica*, 46(11):1843–1851, 2010.

[9] W.W. Hager. Lipschitz continuity for constrained processes. *SIAM Journal on Control and Optimization*, 17(3):321–338, 1979.

[10] W.W. Hager, H. Hou, S. Mohapatra, A.V. Rao, and X.-S. Wang. Convergence rate for a radau hp collocation method applied to constrained optimal control. *Computational Optimization and Applications*, 74(1):275–314, 2019.

[11] W.W. Hager, H. Hou, and A.V. Rao. Convergence rate for a gauss collocation method applied to unconstrained optimal control. *Journal of Optimization Theory and Applications*, 169(3):801–824, 2016.

[12] R.F. Hartl, S.P. Sethi, and R.G. Vickson. fa survey of the maximum principles for optimal control problems with state constraints. *SIAM review*, 37(2):181–218, 1995.

[13] M. Kelly. An Introduction to Trajectory Optimization: How to Do Your Own Direct Collocation. *SIAM Review*, 59(4):849–906, 2017.

[14] D. Liberzon. *Calculus of Variations and Optimal Control Theory: A Concise Introduction*. Princeton University Press, 2012.

[15] H. Maurer and D. Augustin. Sensitivity analysis and real-time control of parametric optimal control problems using boundary value methods. In *Online Optimization of Large Scale Systems*, pages 17–55. Springer, 2001.

[16] V.G. Maz'ya. A new approximation method and it's applications to the calculation of volume potentials, boundary point method. *DFG-Kolloquium*, 16(2):1047–1069, 1991.

[17] V.G. Maz'ya and G. Schmidt. On approximate approximations using Gaussian kernels. *IMA Journal of Numerical Analysis*, 16(1):13–29, 1996.

[18] V.G. Maz'ya and G. Schmidt. On Quasi-interpolation with Non-uniformly Distributed Centers on Domains and Manifolds. *Journal of Approximation Theory*, 110(2):125–145, 2001.

[19] V.G. Maz'ya and G. Schmidt. *Approximate Approximations*. American Mathematical Society, 2007.

[20] H. Mirinejad, T. Inanc, and J.M. Zurada. Radial basis function interpolation and galerkin projection for direct trajectory optimization and costate estimation. *IEEE/CAA Journal of Automatica Sinica*, 8(8):1380–1388, 2021.

[21] Y. Nie, O. Faqir, and E. Kerrigan. ICLOCS2: Try this optimal control problem solver before you try the rest. In *2018 UKACC 12th International Conference on Control*, pages 336–336. IEEE, 2018.

[22] I.M. Ross and M. Karpenko. A Review of Pseudospectral Optimal Control: From Theory to Flight. *Annual Reviews in Control*, 36(2):182–197, 2012.

[23] I.A. Shvartsman and R.B. Vinter. Regularity properties of optimal controls for problems with time-varying state and control constraints. *Nonlinear Analysis: Theory, Methods & Applications*, 65(2):448–474, 2006.

[24] J. Zhu, E. Trélat, and M. Cerf. Geometric optimal control and applications to aerospace. *Pacific Journal of Mathematics for Industry*, 9(1):1–41, 2017.