A Fixed-Time Convergent Distributed Algorithm for Strongly Convex Functions in a Time-Varying Network

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***Abstract*— This paper presents a novel distributed nonlinear protocol for minimizing the sum of convex objective functions in a *fixed* time under time-varying communication topology. In a distributed setting, each node in the network has access only to its private objective function, while exchange of local information, such as, state and gradient values, is permitted between the immediate neighbors. Earlier work in literature considers distributed optimization protocols that achieve convergence of the estimation error in a finite time for static communication topology, or under specific set of initial conditions. This study investigates first such protocol for achieving distributed optimization in a fixed time that is independent of the initial conditions, for time-varying communication topology. Numerical examples corroborate our theoretical analysis.**

I. Introduction

Over the past decade, distributed optimization problems [1] gained considerable attention in the control, optimization and machine learning community. This is primarily due to increase in the size and complexity of datasets, along with privacy concerns and communication constraints among multiple agents. Distributed methods for solving optimization problems find applications in several domains including, but not limited to, distributed multi-agent coordination and estimation in sensor networks [2], formation control [3], model-predictive control [4], economic dispatch [5], resource-allocation [6], cooperative multi-agent motion planning [7] and large-scale machine learning [8] (see [9], [10] for a detailed discussion on applications of distributed optimization in control theory). In a distributed optimization problem, the goal is to cooperatively minimize the sum of local objective functions, each of which is known to only one agent. In particular, the distributed convex optimization problems take the following form:

$$\min_{\mathbf{x}\in\mathbb{R}^d} \quad F(\mathbf{x}) = \sum_{i=1}^{N} f_i(\mathbf{x}), \tag{1}$$

where $F(\cdot)$ is the team objective function, and function $f_i : \mathbb{R}^d \to \mathbb{R}$ represents the local objective function of the i^{th} agent.

Most prior work on distributed optimization primarily concerned with developing discrete-time algorithms that ensure that each agent in the network converges to the optimal point of $F(\cdot)$ or its neighborhood [1], [11], [12]. The dynamical systems perspective of continuous-time optimization has gained much attention in past years for distributed optimization [13], [14], [15], [16]. This viewpoint enables use of tools from Lyapunov theory and differential equations for stability and convergence-rate analysis.

Many practical applications, such as time-critical classification, autonomous distributed systems for surveillance and economic dispatch in power systems, often undergo frequent and severe changes in operating conditions, and thus require quick availability of the optimal solution from any initial condition. Most of the aforementioned work studies dynamical systems with asymptotic or exponential convergence guarantees to the optimal point. In contrast to asymptotic stability or exponential stability that pertains to convergence of the solution to the equilibrium point as time tends to infinity, finite-time stability is a concept that guarantees convergence of solutions in a finite amount of time [17]. Fixed-time stability (FxTS), as defined by the authors in [18], is a stronger notion than finite-time stability (FTS), where the time of convergence does not depend upon the initial condition, and is uniformly bounded for all initial conditions.

In [19], a continuous-time zero-gradient-sum (ZGS) with exponential convergence rate was proposed, which, when combined with a finite-time consensus protocol, was shown to achieve finite-time convergence in [20]. A drawback of ZGS-type algorithms is the requirement of strong convexity of the local objective functions and the requirement of specific initial conditions $\{x_i(0)\}$ for the agents such that $\sum_{i=1}^{N} \nabla f_i(x_i(0)) = 0$. In [21], a novel continuous-time distributed optimization algorithm, based on private (nonuniform) gradient gains, was proposed for convex functions with quadratic growth, and achieves convergence in finite time. A finite-time tracking and consensus-based algorithm was recently proposed in [16], which again achieves convergence in finite time under a time-invariant communication topology. Prior work on multi-agent consensus problems has considered finite- and fixed-time consensus for static topology [22], [23], [24] as well as time-varying topology [25], [26], [27].

In this paper, we consider a general class of nonlinear convex objective function, satisfying the assumption that the team objective is strongly convex, and design a distributed algorithm to compute the optimal solution of (1) when the underlying communication topology is time-varying. To the best of our knowledge, distributed optimization procedures with fixed-time convergence have not been addressed in the

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literature. The authors in [28] consider the distributed optimization problem under the assumption that the individual objective functions belong to a class of quadratic functions. We do not make any such assumption on the objective functions. The authors in [29] propose a fixed-time stable distributed optimization algorithm under the assumption of strong convexity of each local objective function. We relax this assumption in this paper, and require that only the team objective function is strongly convex. The proposed procedure is a distributed tracking and consensus-based algorithm, where both average consensus and tracking are achieved in fixed time. Assumption such as strong convexity of the individual objective functions is also relaxed, and thus the proposed algorithm generalizes to a broader class of distributed optimization problems.

Under the assumption of connectivity being maintained at all times, we show that fixed-time convergence can be guaranteed even when the communication topology varies with time. This aspect helps tackle the scenarios when the environmental/operational conditions lead to changes in the communication topology of the network of the nodes. Previous results on finite- or fixed-time consensus are special cases of the proposed method in this paper, which in addition to achieving fixed-time average consensus, also derives this average value to the optimal solution of the team objective function. The simulation results illustrate that the proposed method gives similar convergence performance as predicted by theory even when simple discretization schemes are implemented, and hence, can be used in practice. The proofs are omitted in this paper, and are available in an extended version available online [30].

The rest of the paper is organized as follows: Section II presents some definitions and lemmas that are useful for designing the fixed-time distributed optimization protocol described in Section III. The protocol is then validated on relevant example scenarios in Section IV. We then conclude our discussion with directions for future work in Section V.

II. Problem Formulation and Preliminaries

We use $\mathbb{R}$ to denote the set of real numbers and $\mathbb{R}_+$ to denote non-negative real numbers. Given a function $f : \mathbb{R}^d \to \mathbb{R}$, its gradient and its Hessian at some point $x \in \mathbb{R}^d$ are denoted by $\nabla f(x)$ and $\nabla^2 f(x)$. Number of agents or nodes is denoted by N. Given $x \in \mathbb{R}^d$, $\|x\|$ denotes the 2-norm of x. $\mathcal{G}(\cdot) = (A(\cdot), \mathcal{V})$ represents an undirected graph with the adjacency matrix $A(t) = [a_{ij}(t)] \in \mathbb{R}^{N\times N}$, $a_{ij} \in \{0, 1\}$ and the set of nodes $\mathcal{V} = \{1, 2, \cdots, N\}$, where $A(t)$ can vary with time t. The set of 1-hop neighbors of node $i \in \mathcal{V}(t)$ is represented by $\mathcal{N}_i(t)$, i.e., $\mathcal{N}_i(t) = \{j \in \mathcal{V} \mid a_{ij}(t) = 1\}$. The second smallest eigenvalue of a matrix is denoted by $\lambda_2(\cdot)$. Finally, we define the function $\text{sign}^\mu : \mathbb{R}^d \to \mathbb{R}^d$ as $\text{sign}^\mu(x) = x\|x\|^{\mu-1}$, for $\mu > 0$ and $\text{sign}(x) \triangleq \text{sign}^0(x)$.

A. Problem statement

Consider the system consisting of N nodes with graph structure $\mathcal{G}(t) = (A(t), \mathcal{V})$ specifying the communication links between the nodes for $t \geq 0$. The objective is to find $x^* \in \mathbf{R}^d$ that solves

$$\begin{aligned} \min_{x_1, x_2, \cdots, x_N} \sum_{i=1}^N f_i(x_i), \\ \text{s.t. } x_1 = x_2 = \cdots = x_N. \end{aligned} \tag{2}$$

In this work, we assume that the minimizer $x^* = x_1^* = x_2^* = \cdots = x_N^*$ for (2) exists and is unique. In contrast to prior work (e.g. [15], [20], [28]), we do not require the private objective functions f_i to be strongly convex, or of a particular functional form. Furthermore, in contrast to [20], where the initial conditions $\{x_i(0)\}$ are required to satisfy ZGS condition, i.e., $\sum_i \nabla f_i(x_i) = 0$, we do not impose any such restrictions. In other words, we show fixed-time convergence for arbitrary initial conditions. We make the following assumptions.

Assumption 1. *The communication topology between the agents at any time instant t is connected and undirected, i.e., the underlying graph $\mathcal{G}(t) = (A(t), \mathcal{V})$ is connected and $A(t)$ is a symmetric matrix for all $t \geq 0$.*

Assumption 2. *Functions f_i are convex, twice differentiable and the Hessian $\nabla^2 F(x) = \sum_{i=1}^N \nabla^2 f_i(x) \succeq kI$, where $k > 0$, for all $x \in \mathbb{R}^d$.*

Assumption 3. *Each node i receives $x_j, \nabla f_j(x_j)$ from each of its neighboring nodes $j \in \mathcal{N}_i$.*

Remark 1. *Assumption 2 states that the team objective function $F(\cdot)$ is strongly-convex with modulus k, and can be easily satisfied even if just one of the objective functions is strongly convex. Assumption 2 also implies that $x = x^*$ is a minimizer if and only if it satisfies $\sum_{i=1}^n \nabla f_i(x^*) = 0$.*

Let $x_i \in \mathbb{R}^d$ represent the state of agent i. We model agent i as a first-order integrator system, given by:

$$\dot{x}_i = u_i, \tag{3}$$

where $u_i \in \mathbb{R}^d$ can be regarded as a control input, that depends upon the states of the agent i, and the states of the neighboring agents $j_1, j_2, \cdots, j_l \in \mathcal{N}_i$. The control input u_i maybe discontinuous, and thus, the solution of (3) are understood in the sense of Filippov [31]. The problem statement is formally given as follows.

Problem 1. *Design u_i for each agent $i \in \mathcal{V}$, such that $x_1 = x_2 = \cdots = x_N = x^*$ is achieved for (3) within a specified fixed time $0 < T < \infty$, for any initial condition $\{x_1(0), x_2(0), \cdots, x_N(0)\}$, where x^* is the minimizer of the team objective function $F = \sum_i f_i$ in (1), i.e., $x_i(t) = x^*$ is achieved for all $i \in \mathcal{V}$, for $t \geq T$.*

B. Overview of FxTS

In this section, we present relevant definitions and results on FxTS. Consider the system:

$$\dot{x}(t) = f(x(t)), \tag{4}$$

where $x \in \mathbb{R}^d$, $f : \mathbb{R}^d \to \mathbb{R}^d$ and $f(0) = 0$. Assume that the solution of (4) exists and is unique. As defined in

[17], the origin is said to be an FTS equilibrium of (4) if it is Lyapunov stable and finite-time convergent, i.e., for all $x(0) \in D \setminus \{0\}$, where D is some open neighborhood of the origin, $\lim_{t \to T(x(0))} x(t) = 0$, where $T(x(0)) < \infty$. The authors in [18] presented the following result for fixed-time stability, where the time of convergence does not depend upon the initial condition, i.e., the settling-time function T does not depend on the initial condition $x(0)$.

Lemma 1 ([18]). Suppose there exists a positive definite continuously differentiable function $V : \mathbb{R}^d \to \mathbb{R}$ for system (4) such that $\dot{V}(x(t)) \le -aV(x(t))^p - bV(x(t))^q$ with $a, b > 0$, $0 < p < 1$ and $q > 1$. Then, the origin of (4) is FxTS, i.e., $x(t) = 0$ for all $t \ge T$, where the settling time T satisfies $T \le \frac{1}{a(1-p)} + \frac{1}{b(q-1)}$.

III. Main Result

Our approach to fixed-time multi-agent distributed optimization is based on first designing a centralized fixed-time protocol that relies upon global information. Then, the quantities in the centralized protocol are estimated in a distributed manner. In summary, the algorithm proceeds by first estimating global quantities (g as defined in (6)) required for the centralized protocol, then driving the agents to reach consensus ($x_i(t) = x(t)$ for all $i \in V$), and finally driving the common trajectory $x(t)$ to the optimal point x^*, all within a fixed time T. Recall that agents are said to have reached consensus on states $\{x_i\}$ if $x_i = x_j$ for all $i, j \in V$. To this end, we define first a novel centralized fixed-time protocol in the following theorem. Note that in the centralized setting, the states of all the agents are driven by the same input g and are initialized to the same starting point. In a distributed setting, this behavior translates to agents having reached consensus and subsequently being driven by a common input (see Remark 2).

Theorem 1 (Centralized fixed-time protocol). Suppose the dynamics of each agent $i \in V$ in the network is given by

$$\dot{x}_i = g, \quad \text{and } x_i(0) = x_j(0) \text{ for all } i, j \in V, \qquad (5)$$

where g is based on global (centralized) information as follows:

$$g = -\left(\sum_{i=1}^N \nabla^2 f_i(x)\right)^{-1}\left(\sum_{i=1}^N \nabla f_i + \mathrm{sign}^{l_1}\left(\sum_{i=1}^N \nabla f_i(x)\right) + \mathrm{sign}^{l_2}\left(\sum_{i=1}^N \nabla f_i(x)\right)\right) \qquad (6)$$

where $l_1 > 1$ and $0 < l_2 < 1$, and $x_i(t) = x(t)$ for each $i \in V$, for all $t \ge 0$. Then the trajectories of all agents converge to the optimal point x^*, i.e., the minimizer of the team objective function (2) in a fixed time $T > 0$.

The proof is based on choosing a Lyapunov candidate $V = \frac{1}{2}\left(\sum_{i=1}^N \nabla f_i(x)\right)^T\left(\sum_{i=1}^N \nabla f_i(x)\right)$, and showing that its time derivatives along the closed-loop trajectories satisfy conditions of Lemma 1.

Remark 2. Theorem 1 represents a centralized protocol for convex optimization of team objective functions. Here, the agents are already in consensus and have access to global information $\sum_{i=1}^N \nabla f_i(x)$. In the distributed setting, agents only have access to their local information, as well as $\nabla f_j(x_j)$ for all $j \in N_i(t)$, and will not be in consensus in the beginning. Below we propose schemes for estimation of global quantities that achieve consensus in fixed time.

For each agent $i \in V$, define $g_i(t)$ as:

$$g_i(t) = -\left(\theta_i(t) + \mathrm{sign}^{l_1}(\theta_i(t)) + \mathrm{sign}^{l_2}(\theta_i(t))\right), \qquad (7)$$

where g_i denotes agent i's estimate of g and $\theta_i : \mathbb{R}_+ \to \mathbb{R}^d$ is the estimates of the global quantities, whose dynamics is given as

$$\dot{\theta}_i(t) = \omega_i(t) + h_i(t), \qquad (8)$$

where $h_i \triangleq \frac{d}{dt}\nabla f_i(x_i(t))$. The input $\omega : \mathbb{R}_+ \to \mathbb{R}^d$, defined as

$$\omega_i = p \sum_{j \in N_i}\left(\mathrm{sign}(\theta_j - \theta_i) + \alpha\,\mathrm{sign}^{\nu_1}(\theta_j - \theta_i) + \beta\,\mathrm{sign}^{\nu_2}(\theta_j - \theta_i)\right), \qquad (9)$$

where $p, \alpha, \beta > 0$, and $0 < \nu_2 < 1 < \nu_1$, helps achieve consensus over the quantities θ_i as shown below. Let $\theta_i(t) = [\theta_{i1}(t), \theta_{i2}(t), \ldots, \theta_{id}(t)]^T$. Note that $\{\theta_i\}$ are updated in a distributed manner.

Assume that $\|h_i - h_j\| \le \rho$ for all $t \ge 0$, for some $\rho > 0$. Although the assumption is somewhat restrictive, it can be easily satisfied if the graph is connected for all time t, the gradients and their partial derivatives are bounded. Many common objective functions, such as quadratic cost functions satisfy this assumption. Under this assumption, we can state the following results.

Lemma 2. Let $\|h_i(t) - h_j(t)\| \le \rho$ for some $\rho > 0$ and all $t > 0$, and the control gain p in (9) satisfies $p > \frac{N-1}{2}\rho$; then there exists a fixed-time $T_1 > 0$, such that for each agent $i \in V$, $\theta_i(t) = \frac{1}{N}\sum_{j=1}^N \theta_j(t)$ for all $t \ge T_1$.

Define $\theta_{ji} \triangleq \theta_j - \theta_i$, and the mean of θ_i's by $\theta_c \triangleq \frac{1}{N}\sum_{j=1}^N \theta_j$. The difference between an agent i's state θ_i and the mean θ_c of all agents' states is denoted by $\tilde{\theta}_i \triangleq \theta_i - \theta_c$. Similarly, $\tilde{\theta}_{ji}$ represents the difference $\tilde{\theta}_j - \tilde{\theta}_i$. The proof is based on choosing a Lyapunov candidate $V = \frac{1}{2}\sum_{i=1}^N \tilde{\theta}_i^T\tilde{\theta}_i$, and showing that its time derivative along the trajectories of θ_i satisfy conditions of Lemma 1.

Theorem 2 (Fixed-time parameter estimation). Let $\theta_i(0) = 0_d$ for each $i \in V$, i.e., agents initialize their local states at origin, and the control gain p in (9) is sufficiently large, more precisely, $p > \frac{N-1}{2}\rho$. Then there exists a fixed-time $0 < T_1 < \infty$ such that $g_i(t) = g_j(t)$ for all $i, j \in V$ and $t \ge T_1$.

Remark 3. Theorem 2 states that if the control gain p is sufficiently large, then the agents estimate the global information $\sum_{i=1}^{N} \nabla f_i(x_i)$ in a distributed manner. Theorem 2 only guarantees that $g_i(t) = g_j(t)$ for all $i,j \in \mathcal{V}$ and $t \geq T_1$. However, in order to employ the centralized fixed-time protocol, agents must additionally reach consensus in their states $\{x_i\}$, so that $g_i(t)$ maps to g for each agent $i \in \mathcal{V}$.

In order to achieve consensus and optimal tracking, we propose the following update rule for each agent $i \in \mathcal{V}$ in the network:

$$u_i = \tilde{u}_i + g_i, \tag{10}$$

where g_i is as described in (7), and $\tilde{u}_i$ is defined as locally averaged signed differences:

$$\tilde{u}_i = q \sum_{j \in \mathcal{N}_i} \Big(\mathrm{sign}(x_j - x_i) + \alpha\, \mathrm{sign}^{\nu_1}(x_j - x_i) + \beta\, \mathrm{sign}^{\nu_2}(x_j - x_i) \Big), \tag{11}$$

where $q, \alpha, \beta > 0$, $\nu_1 > 1$ and $0 < \nu_2 < 1$. The following results establish that the state update rule for each agent proposed in (10) ensures that the agents reach global consensus and optimality in fixed-time.

Theorem 3 (Fixed-time consensus). Under the effect of update law u_i (10) with $\tilde{u}_i$ defined as in (11), and $g_i(t) = g_j(t)$ for all $t \geq T_1$ and $i,j \in \mathcal{V}$, the closed-loop trajectories of (3) converge to a common point $\bar{x}$ for all $i \in \mathcal{V}$ in a fixed time T_2, i.e., $x_i(t) = \bar{x}(t)$ for all $t \geq T_1 + T_2$.

Finally, the following corollary establishes that the agents track optimal point in a fixed-time.

Corollary 1 (Fixed-time distributed optimization). Let each agent $i \in \mathcal{V}$ in the network be driven by the control input u_i (10). Then there exists $T_3 < \infty$ such that the agents track the minimizer of the team objective function within fixed time $T = T_1 + T_2 + T_3$.

There may exist some communication link failures or additions among agents in a network, which results in a time-varying communication topology. We model the underlying graph $\mathcal{G}(t) = (\mathcal{A}(t), \mathcal{V})$ through a switching signal $\sigma : \mathbb{R}_+ \to \Theta$ as $\mathcal{G}(t) = \mathcal{G}_{\sigma(t)}$, where Θ is a finite set consisting of index numbers associated to specific adjacency matrices that satisfy Assumption 1.

Corollary 2 (Time-varying topology). Corollary 1 continues to hold even if communication topology switches among Θ for the multi-agent system described in (3).

The proof follows by defining λ_2 as the minimum of the second smallest eigenvalues of graph Laplacians of the associated adjacency matrices, i.e., $\lambda_2 = \min_{\sigma} \lambda_2(L_{\mathcal{A}(t)})$ and noting that $\lambda_2 > 0$.

The overall fixed-time distributed optimization protocol is described in Algorithm 1. Note that the total time of convergence $T = T_1 + T_2 + T_3$ depends upon the design parameters and is inversely proportional to $p, q, \alpha, \beta, \ldots$. Hence, for a given user-defined time budget $\bar{T}$, one can choose large values of these parameters so that $T \leq \bar{T}$, and hence, convergence can be achieved within user-defined time $\bar{T}$.

Algorithm 1 Fixed-time distributed optimization algorithm.

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procedure FXTS-DISTOPT($(\mathcal{A}, \mathcal{V}), f(\cdot)$)
  For each agent $i \in \mathcal{V}$:
  FxTS Parameter Estimation
  while $t < T_1$, **do**
    Simulate (8) using control law (9)
  end while
  FxTS Consensus
  while $t < T_1 + T_2$ **do**
    Simulate (3) using control law (10)
    Simulate (8) using control law (9)
  end while
  FxTS Optimal Tracking
  while $t < T_1 + T_2 + T_3$ **do**
    Continue simulating (3) using control law (10) with $g_i(t) = g(t)$
  end while
  return x
end procedure

IV. Numerical Examples

In this section, we present numerical examples demonstrating the efficacy of the proposed method. We use semilog-scale to clearly show the variation near the origin, while we show the linear-scale plot in the inset of each figure. Simulation parameters in Theorems 1-3 can be chosen arbitrarily as long as the respective conditions are satisfied. We execute the computation using MATLAB R2018a on a desktop with a 32GB DDR3 RAM and an Intel Xeon E3-1245 processor (3.4 GHz). It should be noted that MATLAB code used for implementations is not fully optimized (eg. loop-based implementation is used over matrix manipulation), however, our approach still highlights accelerated convergence behavior in wall-clock time.

A. Example 1: Distributed Optimization with Heterogeneous Convex Functions and Time-Varying Topology

We present a case study where multiple agents aim to minimize the sum of heterogeneous private functions in fixed-time. A graph consisting of 101 nodes is considered with the local and private objective functions $f_i(x_i)$ described by:

$$f_i(x_i) = \frac{1}{2}(x_i - i)^2 + \frac{1}{4}(x_i - i)^4, \tag{12}$$

so that each f_i is convex for $i \in \{1, 2, \ldots, 101\}$. It can be easily shown that $x^* = \frac{N+1}{2} = 51$. The communication topology switches randomly between the line, ring and star-topologies every 2.5s. For simplicity, we use $\alpha = l_1 = \nu_1 = 1.2$, $\beta = l_2 = \nu_2 = 0.8$, $q = p = 10^{10}$, $\gamma = \delta = 10$ and $\kappa = \mu = 10$ in (11), (6) and (9). With these parameters, we

Fig. 1. Example 1 - The gradient of the objective function $\sum_i \nabla f_i(x_i)$ with time for various initial conditions $x_i(0)$ and $_i(0)$.

Fig. 3. Example 1 - The gradient of the objective function $\sum_i \nabla f_i(x_i)$ with time for various values of $_1 \in [1.05, 1.5]$; $_2 \in [0.5, 0.95]$. The value of $_1$ increases and that of $_2$ decreases from blue to red.

obtain that $T_1 \leq 0.15$, $T_2 \leq 0.15$ and $T_3 = T \leq 10.02$, which implies final time of convergence is $T_c \leq T_1 + T_2 + T_3 = 10.32$.

Figure 1 shows the variation of $\sum_i \nabla f_i(x_i)$ with time for various initial conditions $x_i(0)$ and $_i(0)$. For various initial conditions, $\nabla F(x)$ drops to the value of $1e^{-6}$ within T_c units. Figure 2 plots the maximum $\max_i(|x_i(t) - x^*|)$ with time and shows the convergence of the individual x_i to the optimal point $x^* = 51$ well within T_c units. Figure 3 plots the sum of the gradients for various exponents $_1, _2$. It can be seen that the rate of convergence increases as $_1$ increases, and $_2$ decreases.

Fig. 2. Example 1 - Individual states $x_i(t)$ with time. The states converge to the optimal point x^*.

B. Example 2: Distributed Support Vector Machine

Consider the following linear Support Vector Machine (SVM) example, where functions f_i are given as:

$$f_i(x_i) = \frac{1}{2}\|x_i\|^2 + \sum_{j=1}^{m} \max\{1 - l_{ij}\, x_i^T z_{ij}\, ;0\}. \qquad (13)$$

Here $x_i; z_{ij} \in R^2$, $l_{ij} \in \{-1; 1\}$ represent separating hyperplane parameters of the i^{th} agent, data points allocated to i^{th} agent and corresponding labels, respectively. The objective is to compute the separating hyperplane x_i that separates z_{ij} on the basis of their labels, i.e., to find x_i such that $x_i^T z_{ij} < 0$ if $l_{ij} = -1$ and $x_i^T z_{ij} > 0$ if $l_{ij} = 1$. [1] The vectors z_{ij} are chosen from a random distribution around the line $x = y$, so that the solution, i.e., the separating hyperplane, to the minimization problem $\min \sum_i f_i(x_i)$ is the vector $[1; -1]$. In this case, we consider a network consisting of 5 nodes connected in a line graph with $m = 100$ randomly distributed points per agent. Figure 4 shows the distribution of z_{ij} symmetrically around the line $x = y$.

For this case, the parameters were set to $_1 = l_1 = _1 = 1.2$, $_2 = l_2 = _2 = 0.8$, $q = p = 50$, $= = = 1$; $= 10$. With these parameter values, we obtain that $T_1; T_2 \leq 0.3$ and $T_3 = T \leq 10.02$, which implies final time of convergence satisfies $T_c \leq T_1 + T_2 + T_3 = 10.62$ units.

Fig. 4. Distribution of points z_{ij} around the line $x = y$ (red dotted line). Blue and red stars denote the points corresponding to $l_{ij} = -1$ and $l_{ij} = 1$, respectively.

Figure 5 illustrates the variation of $\nabla F(x) = \sum_i \nabla f_i(x)$. Figure 6 plots the convergence behavior of the state error $x_i - x^*$. It is clear from the figures that the convergence of the proposed algorithm is superlinear, and that it achieves convergence (up to discretization precision) in a fixed time independent of the initial conditions.

V. Conclusions

In this paper, we presented a scheme to solve a distributed convex optimization problem for continuous time multi-agent

[1] Since the proposed method assumes that the functions f_i are twice differentiable, we use function $g(0; a) = \frac{1}{} \log(1 + e^{a})$ with large values of to smoothly approximate $\max\{0; a\}$.