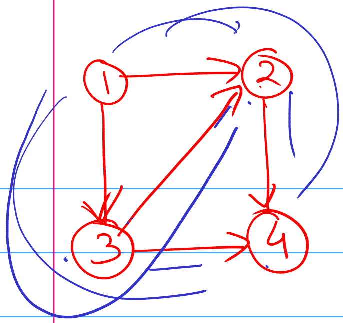
* 

Not strongly connected.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad Ax = \begin{bmatrix} 0 \\ x_1 + x_3 \\ x_1 \\ x_2 + x_3 \end{bmatrix}$$

Not symmetric

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix} \qquad A^2 x = \begin{bmatrix} 0 \\ x_1 \\ 0 \\ 2x_1 + x_3 \end{bmatrix}$$

$A^k x$ } Graph convolutional neural networks

* Consensus: It amounts to all nodes converging to a common value.

* Average Consensus: Consensus + Avg. of all initial values.

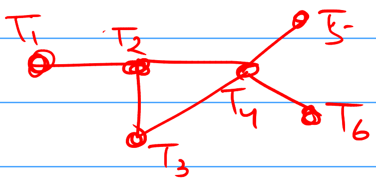
$$\frac{x_1(0) + \cdots + x_n(0)}{n}$$

Sufficient

Conditions under which consensus is achieved.

" " " avg. " " "

* Max consensus or Min consensus:



$\max\{T_1, T_2, \ldots, T_6\}$

Agent i $\quad T_i(k+1) = \max\{T_i(k), T_j(k)\}$

$\forall j \in \mathcal{N}_i$

If d is the graph diameter, then this algorithm converges in d-steps.

* Opinion dynamics:

French-Harary-De Groot Opinion Dynamics:

$$\boxed{p_i(t+1) = \sum_{j=1}^{n} a_{ij}\, p_j(t)}$$

s.t $\sum_{j=1}^{n} a_{ij} = 1 \;\; \forall i$

$p(t+1) = A\,p(t)$

and $a_{ij} \geq 0$

Adjancency matrix

a_{ii} : relative importance to agent i's own belief

A is row-stochastic $\Rightarrow$ row sum is 1.

Q: Does row-stochastic A guarantee consensus? ✓

Q: " " " " average consensus? X

$p(t+1) = A^{t+1}\,p(0)$

$A\,\mathbb{1}_n = \mathbb{1}_n$

1 is an eigenvalue of A with eigenvector $\mathbb{1}_n$.

* Not all A that are row-stochastic will lead to average consensus.

$$x(k+1) = A\,x(k)$$
$$= A^{k+1}\,x(0)$$

Want $\lim_{k\to\infty} x(k) \longrightarrow \dfrac{\sum_{i=1}^{n} x_i(0)}{n}$

* Refresher on Linear Algebra:

Similarity transformation $\boxed{A = P J P^{-1}}$

Square matrices A and J are similar if they can be related using above transformation.

* If J is a diagonal matrix, then we say that A is diagonalizable.

P, J $\qquad A \rightarrow (\lambda_1, \vec{v}_1), \ldots, (\lambda_n, v_n)$

$$\left.\begin{aligned} Av_1 &= \lambda_1 v_1 \\ Av_2 &= \lambda_2 v_2 \\ &\vdots \\ Av_n &= \lambda_n v_n \end{aligned}\right\} \quad A[v_1 \; v_2 \; \cdots \; v_n] = \underbrace{[v_1 \; v_2 \; \cdots \; v_n]}_{P} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & & \cdots & \lambda_n \end{bmatrix}}_{J}$$

A has distinct (simple) eigenvalues.

$$A = PJP^{-1}$$

Fact 1: A real symmetric matrix has real eigenvalues and is also diagonalizable.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \{1\} \qquad N(I-I)$$

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$(\lambda-1)(\lambda-2)(\lambda-4)^2 = 0$

$(A - 4I)p_3 = 0 \rightarrow p_3$ is an eigenvector

$(A-4I)p_4 = p_3$

p_4 generalized eigenvector

$(A-4I)^2 p_4 = 0$

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad J_{2\times 2} = \lambda I_{2\times 2} + N$$

* For any row-stochastic matrix $\rightarrow$ Max eigenvalue is 1, and if the graph is strongly connected, then 1 is also a simple eigenvalue.

* **Fact 2:** 1 is an eigenvalue of any row-stochastic matrix.

* **Fact 3:** No other eigenvalue is more than 1 for row stochastic matrix.

Proof: $\exists\ \lambda > 1$, which is also the eigenvalue of row stochastic matrix A with v being the eigenvector.

$$Av = \lambda v \qquad\qquad \lambda v_i > v_i.$$

$$i \in \arg\max_{j \in [n]} |v_j|$$

$$(Av)_i \leq v_i \qquad [\text{Contradiction}]$$

* **Fact 4:** If the underlying graph is strongly connected, then 1 is also a simple eigenvalue.

* **Theorem:** Let $A \in \mathbb{R}^{n \times n}$, $n \geq 2$, be a non-negative matrix with dominant eigenvalue λ and the right and left eigenvectors are v and w. s.t $v^T w = 1$. If λ is simple and strictly larger in magnitude than all other eigenvalues, then we have:

$$\lim_{k \to \infty} \frac{A^k}{\lambda^k} = v w^T$$

* Application of above Theorem in the context of avg. consensus

  * A is row-stochastic $\qquad \lambda = 1$, $\quad v = \mathbb{1}_n$
  * If the underlying graph is connected $\Rightarrow \lambda = 1$ is simple.
  * If A is symmetric $\qquad w = \frac{\mathbb{1}_n}{n}$

$$\Rightarrow \lim_{k\to\infty} A^k = \frac{1 1^T}{n}$$

$$x(k+1) = A^{k+1} x(0)$$

$$\lim_{k\to\infty} x(k+1) = \frac{1 \; 1^T x(0)}{n}$$

$$= \frac{1}{n}\begin{bmatrix} \sum_{i=1}^{n} x_i(0) \\ \vdots \\ \vdots \end{bmatrix} \Longrightarrow \text{Average consensus}$$

Sufficient condⁿ:

- ✓ A is row-stochastic
- ✓ A is symmetric
- ✓ Underlying graph is connected

$\Rightarrow$ Average consensus

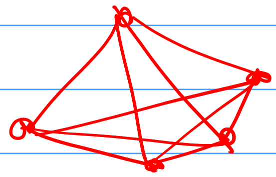
* Line graph:

n nodes

edges $= n-1$

diameter $= n-1$

* Complete graph:



edges $= \binom{n}{2}$

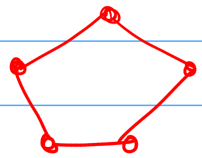
diameter $= 1$

↑ Consensus is faster

↓ Communication bandwidth requirement

* Static Exponential Graphs

* Ring graph:



diameter is nearly half of that of line graph.

* Some important results:

(1)
$$\sum_{i=1}^{N} \sum_{j \in N_i} \operatorname{sign}(x_i - x_j) = 0$$

for undirected, unweighted graph.

$f(x_i - x_j)$

↳ f is an odd function

Proof:
$$T = \sum_{i=1}^{N} \sum_{j \in N_i} \operatorname{sign}(x_i - x_j)$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \operatorname{sign}(x_i - x_j)$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ji} \operatorname{sign}(x_j - x_i)$$

$$= - \sum_{i,j=1}^{N} a_{ij} \cdot \operatorname{sign}(x_i - x_j) = -T$$

$\Rightarrow T = 0$ [Hence Proved]

(2) $$\sum_{i,j=1}^{N} a_{ij}\, e_i^T w(x_{ij}) = \frac{1}{2} \sum_{i,j=1}^{N} a_{ij}\, e_{ij}^T w(x_{ij})$$

where, w is an odd function,

i.e. $w(-x) = -w(x)$

$e_{ij} := e_i - e_j$

$x_{ij} := x_i - x_j$

$$\sum_{i=1}^{N} \sum_{j \in N_i} e_i^T w(x_{ij}).$$

* **Proof**:

$$\sum_{i,j=1}^{N} a_{ij}\, e_i^T w(x_{ij}) = -\sum_{i,j=1}^{N} a_{ij}\, e_i^T w(x_{ji})$$

$$= -\sum_{i,j=1}^{N} a_{ji}\, e_j^T w(x_{ij})$$

$$= -\sum_{i,j=1}^{N} a_{ij}\, e_j^T w(x_{ij})$$

$$2\sum_{i,j=1}^{N} a_{ij}\, e_i^T w(x_{ij}) = \sum_{i,j=1}^{N} a_{ij} (e_i - e_j)^T w(x_{ij})$$

$$= \sum_{i,j=1}^{N} a_{ij}\, e_{ij}^T w(x_{ij})$$

$$\sum_{i,j=1}^{N} a_{ij}\, e_i^T w(x_{ij}) = \frac{1}{2} \sum_{i,j=1}^{N} a_{ij}\, e_{ij}^T w(x_{ij})$$

—x—x—x—