* Discrete Consensus algorithm:

$$x(k+1) = A\,x(k)$$

$$\hookrightarrow \quad A_{ij} = \begin{cases} \dfrac{1}{1+\max\{d_i, d_j\}} & \text{if } i \neq j \\ 1 - \sum_{j \neq i} A_{ij} & \text{if } i = j \end{cases}$$

* Simple consensus algorithm:

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j)$$

$$\dot{x} = -Lx \rightarrow L \text{ is graph Laplacian}$$

* FxTS consensus algorithm:

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} \left[\mathrm{sgn}^{\mu_1}(x_i - x_j) + \mathrm{sgn}^{\mu_2}(x_i - x_j) \right]$$

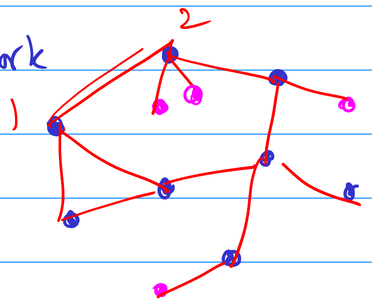
$$\mathrm{sgn}^{\mu}(x) := x \, \|x\|^{\mu - 1}$$

$$\hookrightarrow e_i \qquad e_i \cdot \mathrm{sgn}^{\nu}(e_i) = \|\tau_i\|^2 \|e_i\|^{\nu - 1}$$

—— X —— X —— X —— X —— X —— X ——

Distributed Economic Dispatch Problem (EDP)

Power Network



- Generator buses
- Load buses.

$P_{tot} \rightarrow$ Total load demand at any given time.

- Each generator has its own private cost coefficient.

$$C_i := C_i(P_i) = \alpha_i P_i^2 + \beta_i P_i + \gamma_i$$

$\alpha_i, \beta_i, \gamma_i$ are private cost coefficients.

$$\sum_{i=1}^{N} P_i = P_{tot} \quad \text{(there are } N \text{ generators in the network)}$$

- Generator Capacity constraints.

$$P_{i,min} \le P_i \le P_{i,max}$$

$$\min_{\{P_i\}} \sum_{i=1}^{N} \alpha_i P_i^2 + \beta_i P_i + \gamma_i$$

$$\text{s.t.} \quad \sum_{i=1}^{N} P_i = P_{tot}$$

$\longrightarrow$ Uncapacitated EDP

$$\text{and} \quad P_{i,min} \le P_i \le P_{i,max}$$

$\longrightarrow$ Capacitated EDP

Goal: Develop fixed-time convergent optimization algorithm for solving the capacitated EDP.

Aside:

$$\sum_{i=1}^{N} f_i(x_i)$$

$$\text{s.t} \quad x_1 = x_2 = \dots = x_N$$

★ Uncapacitated EDP:

$$L(\{P_i\},\lambda) = \sum_{i=1}^{N} \alpha_i P_i^2 + \beta_i P_i + \gamma_i + \lambda\left(P_{tot} - \sum_{i=1}^{N} P_i\right)$$

↳ Lagrangian

$$\frac{\partial L}{\partial P_i} = 0 \Rightarrow 2\alpha_i P_i^* + \beta_i - \lambda^* = 0$$

$$\boxed{\lambda^* = 2\alpha_i P_i^* + \beta_i} \quad \text{— (1)}$$

$$P_i^* = \frac{\lambda^* - \beta_i}{2\alpha_i} \quad \text{— (2)}$$

$$\frac{\lambda^*}{2\alpha_i} = P_i^* + \frac{\beta_i}{2\alpha_i}$$

$$\lambda^*\left(\sum_{i=1}^{N} \frac{1}{2\alpha_i}\right) = P_{tot} + \sum_{i=1}^{N} \frac{\beta_i}{2\alpha_i}$$

$$\boxed{\lambda^*_{un} = \frac{P_{tot} + \sum_{i=1}^{N} \frac{\beta_i}{2\alpha_i}}{\sum_{i=1}^{N} \frac{1}{2\alpha_i}}}$$

Algorithm for Uncapacitated EDP:

- We need to keep couple of objectives in mind.
  - We need to run consensus on λ.
  - $P_i = \frac{\lambda_i - \beta_i}{2\alpha_i}$ → Need to ensure this.

Every generator runs this algorithm

$$\dot{P}_i(t) = \sum_{j \in N_i} \left[\mathrm{sgn}^{\mu_1}(\lambda_j(t) - \lambda_i(t)) + \mathrm{sgn}^{\mu_2}(\lambda_j(t) - \lambda_i(t)) \right]$$

$$\frac{\dot{\lambda}_i(t)}{2\alpha_i} = \dot{P}_i(t) + \underbrace{\left[\mathrm{sgn}^{\nu_1}\left(P_i - \frac{\lambda_i - \beta_i}{2\alpha_i}\right) + \mathrm{sgn}^{\nu_2}\left(P_i - \frac{\lambda_i - \beta_i}{2\alpha_i}\right) \right]}_{T_1}$$

$T_2 > T_1$

$$\sum \dot{P}_i(t) = 0 \Rightarrow \sum P_i(t) = \sum_i P_i(0) = P_{tot}$$

Proof of fixed-time convergence of the above algorithm:

We are first going to show that the second ODE, equilibrium is reached in a fixed time.

$$e_i = P_i - \left(\frac{\lambda_i - \beta_i}{2\alpha_i}\right)$$

$$\dot{e}_i = \dot{P}_i - \frac{\dot{\lambda}_i}{2\alpha_i}$$

$$= - \left[\mathrm{sgn}^{\nu_1}\left(P_i - \frac{\lambda_i - \beta_i}{2\alpha_i}\right) + \mathrm{sgn}^{\nu_2}\left(P_i - \frac{\lambda_i - \beta_i}{2\alpha_i}\right) \right]$$

$$\dot{e}_i = - \mathrm{sgn}^{\nu_1}(e_i) - \mathrm{sgn}^{\nu_2}(e_i)$$

$$V = \frac{1}{2} \sum_{i=1}^{N} e_i^2$$

$$\dot{V} = \sum_{i=1}^{N} e_i \dot{e}_i = - \sum_{i=1}^{N} e_i \,\mathrm{sgn}^{\nu_1}(e_i) - \sum_{i=1}^{N} e_i \,\mathrm{sgn}^{\nu_2}(e_i)$$

$$\Rightarrow \dot{V} = -\sum_{i=1}^{N} |e_i|^{1+\nu_1} - \sum_{i=1}^{N} |e_i|^{1+\nu_2}$$

$$= -\sum_{i=1}^{N} (e_i^2)^{\frac{1+\nu_1}{2}} - \sum_{i=1}^{N} (e_i^2)^{\frac{1+\nu_2}{2}}$$

$\nu_1 \in (0,1)$ $\quad \frac{1+\nu_1}{2} \in (0,1)$

$\nu_2 > 1$

$\Rightarrow \frac{1+\nu_2}{2} > 1$

$z_i > 0$

$$\begin{cases} \sum_{i=1}^{N} z_i^{p} \geq \left(\sum_{i=1}^{N} z_i\right)^{p} & \text{if } p \in (0,1) \\ \sum_{i=1}^{N} z_i^{p} \geq N^{1-p} \left(\sum_{i=1}^{N} z_i\right)^{p} & \text{if } p > 1 \end{cases}$$

$$\Rightarrow \dot{V} \leq -\left(\sum_{i=1}^{N} e_i^2\right)^{\frac{1+\nu_1}{2}} - \left(\sum_{i=1}^{N} e_i^2\right)^{\frac{1+\nu_2}{2}} N^{1-\frac{1+\nu_2}{2}}$$

$$\Rightarrow \boxed{\dot{V} \leq -(2V)^{\frac{1+\nu_1}{2}} - N^{\frac{1-\nu_2}{2}} (2V)^{\frac{1+\nu_2}{2}}}$$

$\Rightarrow$ Convergence in fixed-time

i.e, $\exists\, T_1 < \infty$, s.t. 2^{nd} ODE converges to its equilibrium in a fixed-time $t \leq T_1$

$\Rightarrow$ After time T_1, we have

$$\frac{\dot{\lambda}_i}{2\alpha_i} = -\sum_{j \in N_i} \left[\mathrm{sgn}^{\mu_1}(\lambda_i - \lambda_j) + \mathrm{sgn}^{\mu_2}(\lambda_i - \lambda_j)\right]$$

Consensus on λ_i

In this case $\quad \lambda_c = \frac{\Gamma}{N} \sum_{i=1}^{N} \frac{\lambda_i}{2\alpha_i}$

where, $\Gamma = \dfrac{1}{\sum_{i=1}^{N} \frac{1}{2\alpha_i}}$

$$\tilde{\lambda}_i = \lambda_i - \lambda_c$$

$$\dot{\tilde{\lambda}}_i = \dot{\lambda}_i$$

$$V = \frac{1}{2} \sum_{i=1}^{N} \frac{\tilde{\lambda}_i^2}{2\alpha_i} \longrightarrow \dot{V} \leq -c_1 V^{\frac{1+\mu_1}{2}} - c_2 V^{\frac{1+\mu_2}{2}}$$

λ is also called incremental cost variable and there is a popular discretized algorithm k/a Incremental Cost Consensus (ICC) used to solve uncapacitated EDP.

Capacitated EDP (Distributed)

$$\min_{\{P_i\}} \sum_{i=1}^{N} \alpha_i P_i^2 + \beta_i P_i + \gamma_i$$

$$\text{s.t.} \quad \sum_{i=1}^{N} P_i = P_{tot} \checkmark$$

$$\text{and} \quad \begin{array}{l} P_i \leq P_{i,max} \\ P_i \geq P_{i,min} \end{array} \quad \forall i \in \{1, \dots, N\}$$

$$L(\{P_i\}, \lambda, \{r_i\}, \{s_i\}) = \sum_{i=1}^{N} \alpha_i P_i^2 + \beta_i P_i + \gamma_i + \lambda\left(P_{tot} - \sum_{i=1}^{N} P_i\right) \\ \left.\begin{array}{l} + \sum_{i=1}^{N} r_i (P_i - P_{i,max}) \\ + \sum_{i=1}^{N} s_i (P_{i,min} - P_i) \end{array}\right\} r_i, s_i \geq 0$$

$$\frac{\partial L}{\partial P_i} = 0 \Rightarrow 2\alpha_i P_i + \beta_i - \lambda + r_i - s_i = 0$$

$$\lambda = 2\alpha_i P_i + \beta_i + r_i - s_i$$

Complementary slackness: $r_i (P_i - P_{i,max}) = 0 \quad \forall i$

$s_i (P_{i,min} - P_i) = 0 \quad \forall i$

If $P_i = P_{i,min}$, then

$$\lambda < 2\alpha_i P_i + \beta_i$$

and if $P_i = P_{i,max}$, then

$$\lambda > 2\alpha_i P_i + \beta_i$$

$$\lambda^* = \begin{cases} 2\alpha_i P_i^* + \beta_i & \text{if } P_i^* \in (P_{i,min}, P_{i,max}) \\ > 2\alpha_i P_i^* + \beta_i & \text{if } P_i^* = P_{i,max} \\ < 2\alpha_i P_i^* + \beta_i & \text{if } P_i^* = P_{i,min} \end{cases}$$

Θ: set of generators for which inequality constraints are active. $P_i^* = P_{i,min}$ or $P_i^* = P_{i,max}$

$$\frac{\frac{1}{N}\sum_{i\in\Theta}\left(\quad \right)}{\frac{1}{N}\sum_{i\notin\Theta}\frac{1}{2\alpha_i}}$$

$$\boxed{\lambda_{cap}^* = \lambda_{un}^* + \frac{\sum_{i\in\Theta}\left(\frac{\lambda_{un}^* - 2\alpha_i P_i - \beta_i}{2\alpha_i}\right)}{\sum_{i\notin\Theta}\frac{1}{2\alpha_i}}}$$

Algorithm for Capacitated EDP

* $\Theta = \phi$
* Solve uncapacitated EDP $\rightarrow \{\lambda_{un}^*, \{P_i^*\}\}$
* $\lambda^* \Leftarrow \lambda_{un}^*$
* while Generation constraint violation.
  - $\Omega \triangleq \{i \notin \Theta : (P_i < P_{i,min}) \text{ or } (P_i > P_{i,max})\}$

    $\Theta \Leftarrow \Theta \cup \Omega$
  - Calculate optimal dispatch

    $$P_i \Leftarrow \begin{cases} \frac{\lambda_i - \beta_i}{2\alpha_i}, & \text{if } i \notin \Theta \\ P_{i,max} \text{ or } P_{i,min} & \text{if } i \in \Theta \end{cases}$$
  - $$(y_i(0), z_i(0)) \Leftarrow \begin{cases} \left(\frac{\lambda - 2\alpha_i P_i - \beta_i}{2\alpha_i}, 0\right) & i \in \Theta \\ \left(0, \frac{1}{2\alpha_i}\right) & i \notin \Theta \end{cases}$$

- Run avg. consensus on $\{y_i\}$ and $\{z_i\}$
- $\lambda^x \leftarrow \lambda^x + \frac{y_c}{z_c}$

end while