Outline

- ↳ Optimization Problems
- ↳ Convex sets and Convex functions
- ↳ Operations that preserve convexity
- ↳ 1^{st} & 2^{nd} order conditions for convexity
- ↳ Strictly convex functions + Strongly convex functions

Optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x) \quad \to \text{Unconstrained optimization problem}$$

$\to x_1^2 + x_2^2$

Objective function

$$\text{s.t.} \quad h_i(x) \le 0 \quad \forall i \in \{1, 2, \ldots, m\}$$

$$l_j(x) = 0 \quad \forall j \in \{1, 2, \ldots, r\}$$

Constrained optimization problem

x: decision/optimization variable $\quad x = [x_1 \; x_2 \; \ldots \; x_n]$

Convex optimization problem $\to$ f is convex function

h_i's are convex functions

l_j's are linear equalities

* If $\text{dom } f = \mathbb{R}^n$ $\to$ Feasible sol^n set

$$\mathcal{X} = \{ x \in \mathbb{R}^n : h_i(x) \le 0, \; l_j(x) = 0 \;\; \forall i, j \}$$

↳ 1^{st} order primal/dual methods

2^{nd} order methods

* Ex: ℓ_1-regularized least squares problem

$$y = Mx + v$$

v → noise $\in \mathbb{R}^s$ $(s << n)$

M → Measurement matrix $\in \mathbb{R}^{s\times n}$

$y \in \mathbb{R}^s$ (Measured signal)

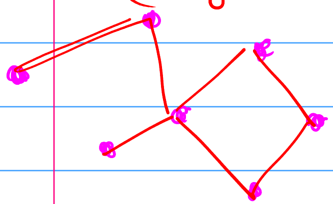
x → unknown signal $\in \mathbb{R}^n$

$$\min_{x\in\mathbb{R}^n} \|y - Mx\|^2 + \|x\|_1$$

$\|x\|_1$ → ensures sparsity

↳ compressive sensing, image processing

(M_i, y_i)



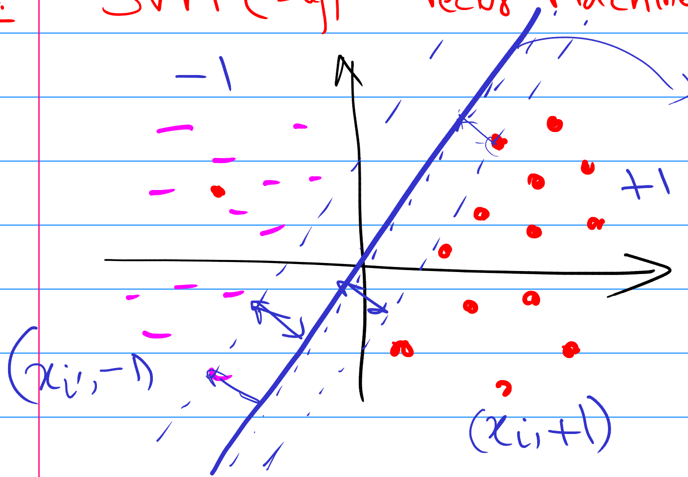
$$\min_{x\in\mathbb{R}^n} \sum_{i=1}^{m} \|y_i - M_i x\|^2 + \|x\|_1$$

(M_i, y_i) are not known to central entity.

$$\min_{x_1, x_2, \dots x_m \in \mathbb{R}^n} \sum_{i=1}^{m} \|y_i - M_i x_i\|^2 + \|x_i\|_1$$

$$\text{st } (x_1 = x_2 = \dots = x_m)$$

Ex: SVM (Support Vector Machines)



$w^T x + b = 0$

Separating hyperplane

$$\begin{cases} w^T x_i + b > 0 \Rightarrow y_i = +1 \\ w^T x_i + b < 0 \Rightarrow y_i = -1 \end{cases}$$

$$d_i = \frac{|w^T x_i + b|}{\|w\|}$$

$$d = \min_i \frac{|w^T x_i + b|}{\|w\|}$$

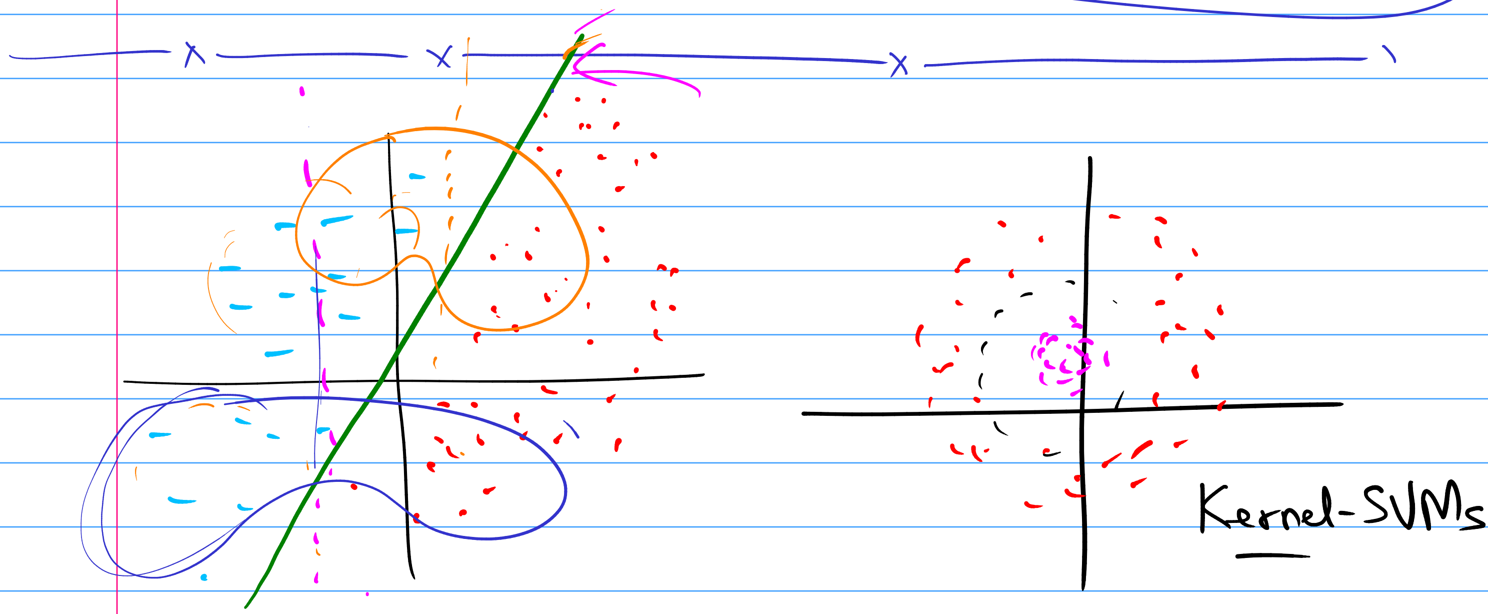
$d\|w\| = 1$ ← fixes scale

$$\max_{\{w,b\}} d = \max_{\{w,b\}} \frac{1}{\|w\|} = \min_{\{w,b\}} \frac{1}{2}\|w\|^2$$

$$\text{st. } y_i(w^T x_i + b) \ge 1$$

$$\boxed{\min_{\{w,b\}} \frac{1}{2}\|w\|^2 \quad \text{s.t.} \quad y_i(w^T x_i + b) \geq 1 \quad \forall i \in \{1, 2, \dots, m\}}$$

(x_i, y_i)



* Convex sets and Convex Functions

$x, y \in \mathbb{R}^n$

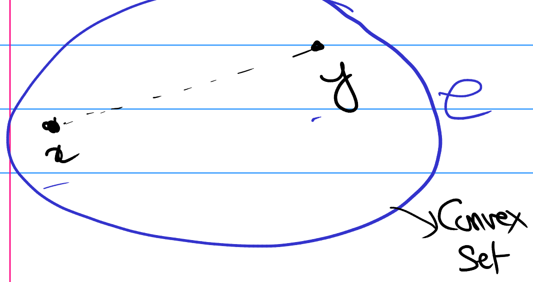
Convex combination is of the form

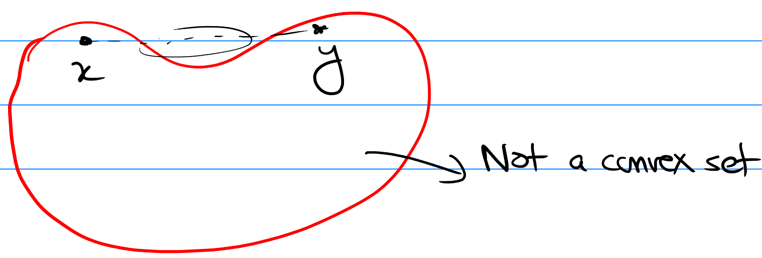
$$z = \theta x + (1-\theta) y \qquad \forall \theta \in [0,1]$$

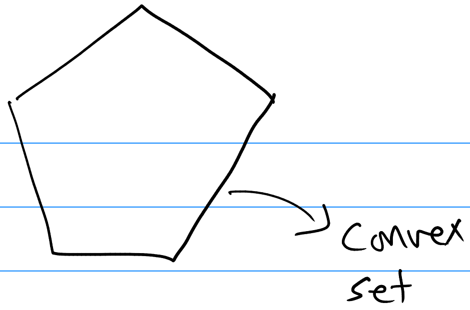
Linear combination

$$\theta_1 x + \theta_2 y$$

$x, y \in \mathcal{C}$, such that their convex combination also lies in $\mathcal{C}$, then $\mathcal{C}$ is a convex set.





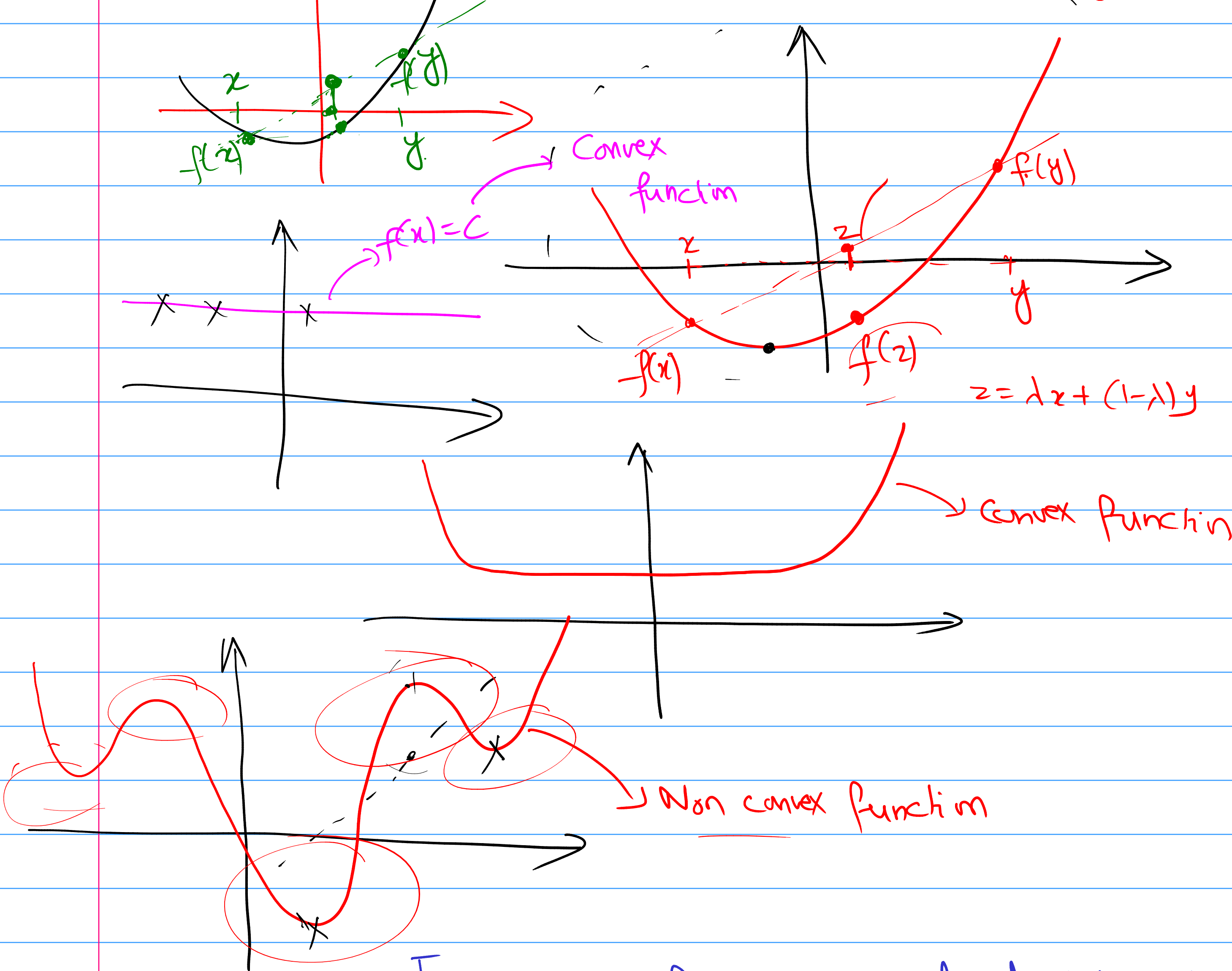


$\mathcal{C} = [1,2) \cup [5,6]$
Not a convex set

* Convex functions: A function f is said to be convex if

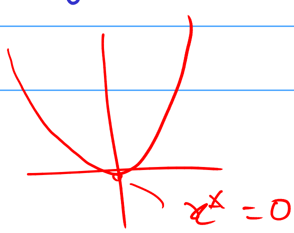
$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

$\forall\, \lambda \in [0,1]$ and $\forall\, x, y \in \mathrm{dom}(f)$

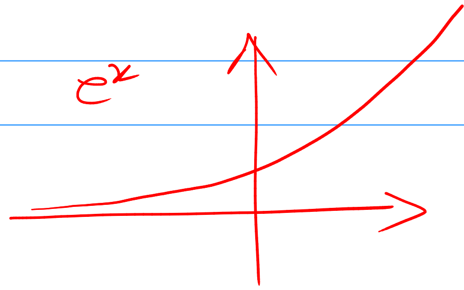


For a convex function, every local minima is also a global minima.

Ex: (a) $f(x) = \frac{1}{2}x^2$

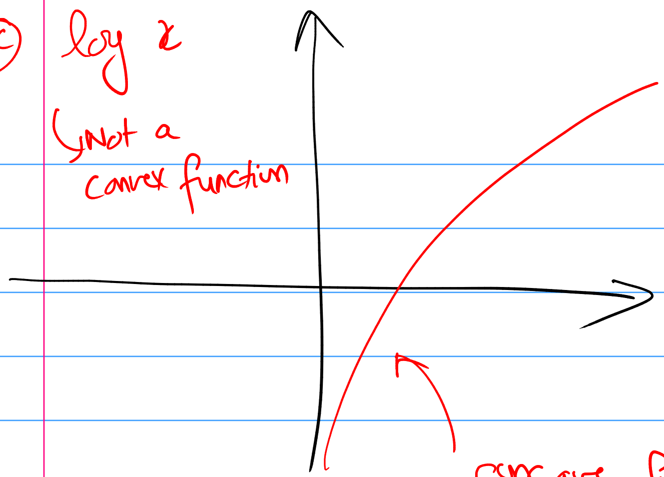


(b) e^x



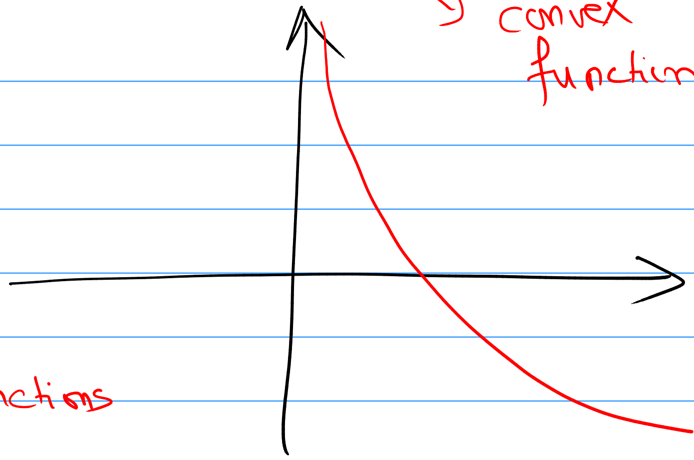
ⓒ $\log x$

↳ Not a convex function



concave functions

ⓓ $-\log x$ → convex function



f is a concave function $\Longleftrightarrow$ $-f$ is a convex function.

* Operations that preserve convexity:

Γ : Class of all convex functions.

① Non-negative weighted sum: $f_1, f_2 \in \Gamma$

$w_1, w_2 \geq 0$

$\Rightarrow w_1 f_1 + w_2 f_2 \in \Gamma$

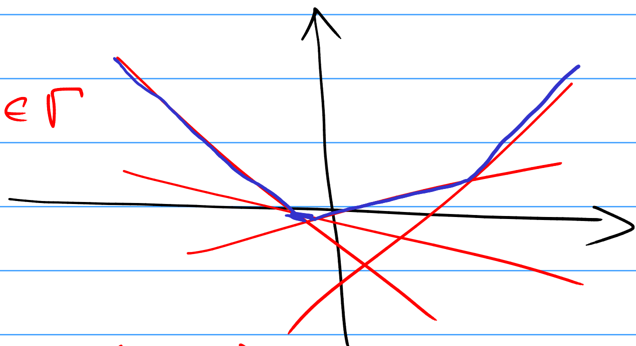
① l_1-penalty

$\|x\|_1 = \sum_{i=1}^{n} |x_i|$ → convex function

② $\frac{1}{2}x^2 + \frac{1}{4}x^4$

② Pointwise max: $f_1, f_2 \in \Gamma$

$\Rightarrow \max\{f_1, f_2\} \in \Gamma$



③ Affine transformation.

$f(x)$ is convex, then $f(Ax+b)$ is also convex

① $\frac{1}{2}\|x\|^2 \Longrightarrow \frac{1}{2}\|Ax-b\|^2$

② Log-barrier function $-\log(b-a^Tx)$ is convex

$a^Tx \leq b$

———x———x———x———

First order condition for convexity -

Assume f is continuously differentiable $\nabla f(x)$ exist

$$\boxed{f \text{ is convex} \iff \text{dom } f \text{ is convex and } f(y) \geq f(x) + \nabla f(x)^T (y-x)}$$

$\Rightarrow$: $f(x+\lambda(y-x)) = f(\lambda y + (1-\lambda)x) \quad \forall \lambda \in [0,1]$

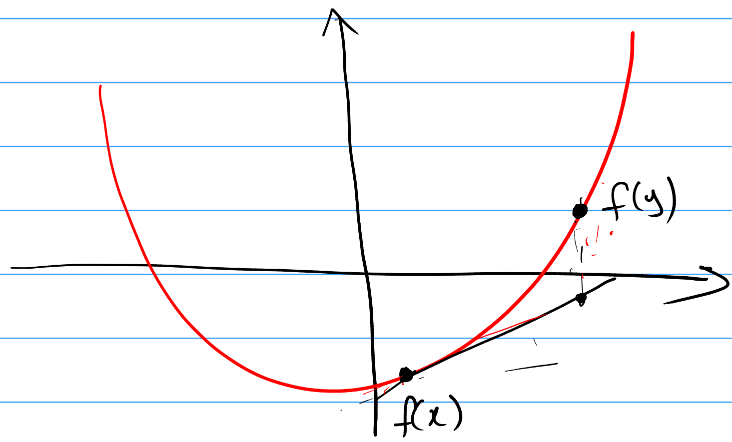
$\leq \lambda f(y) + (1-\lambda) f(x)$ [From def^n of convexity]

$$f(x+\lambda(y-x)) - f(x) \leq \lambda (f(y) - f(x))$$

$$\lim_{\lambda \to 0} \frac{f(x+\lambda(y-x)) - f(x)}{\lambda} \leq f(y) - f(x)$$

$$\nabla f(x)^T (y-x) \leq f(y) - f(x)$$

or $\boxed{f(y) \geq f(x) + \nabla f(x)^T (y-x)}$ ∎



* Second order condition for convexity:

Assume f is twice continuously differentiable.

$$\boxed{f \text{ is convex} \iff \nabla^2 f \text{ is positive semidefinite} + \text{dom } f \text{ is convex}}$$

$f(x) = \frac{1}{2}x^2$ $\quad f'' \geq 0$ $\quad x^T \nabla^2 f(x) x \geq 0 \quad \forall x$

$f''(x) = 1 > 0$ $\quad f(x) = \frac{1}{4}x^4$

$f''(x) = 3x^2 \geq 0$