* GD: $X_{k+1} = X_k - \eta \nabla F(x_k)$

$$\lim_{\eta \to 0} \frac{X_{k+1} - X_k}{\eta} = \dot{X} \quad \Big\} \quad \boxed{\dot{X} = -\nabla F(X)}$$

↳ Gradient Flows

$\dot{X} = 0 \Rightarrow \nabla F(X) = 0$

↳ 1st order condition of optimality.

$$\dot{x} = -\frac{\nabla f(x)}{\|\nabla f(x)\| + \varepsilon}$$

Insights

$$x_{k+1} = x_k - \eta \frac{\nabla f(x_k)}{\|\nabla f(x_k)\| + \varepsilon}$$

$\dot{X} = -\nabla f(X) = 0$

$X = X^* = X_e$

$\nabla f(X^*) = 0$

x^* is an equilibrium point.

Stability of equilibrium

* Stability is a property of equilibria. (and not of dynamical systems)

* Dynamical system:

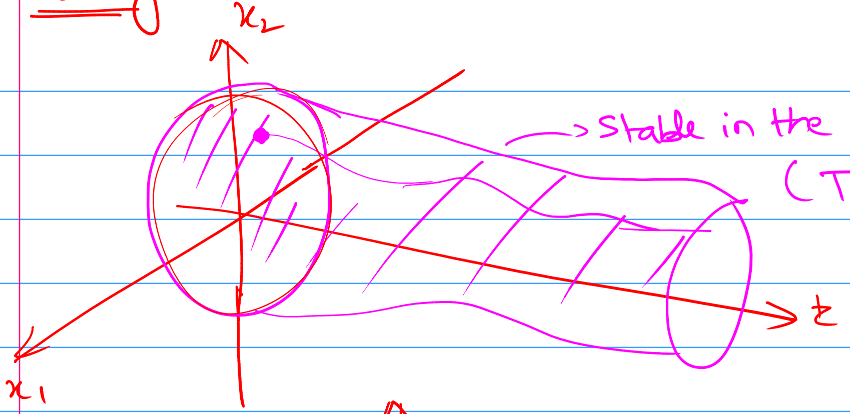
$\dot{x} = g(t, x)$ → Time-varying system → Non-autonomous system

$\dot{x} = g(x)$ → Time-invariant system → Autonomous system

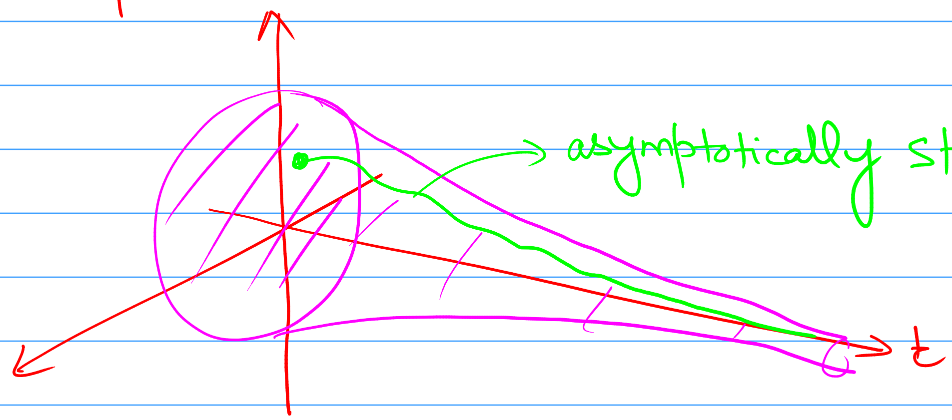
$g(x_e) = 0$ → x_e is the equilibrium

WLOG, we assume $x_e = 0$.

* Stability:



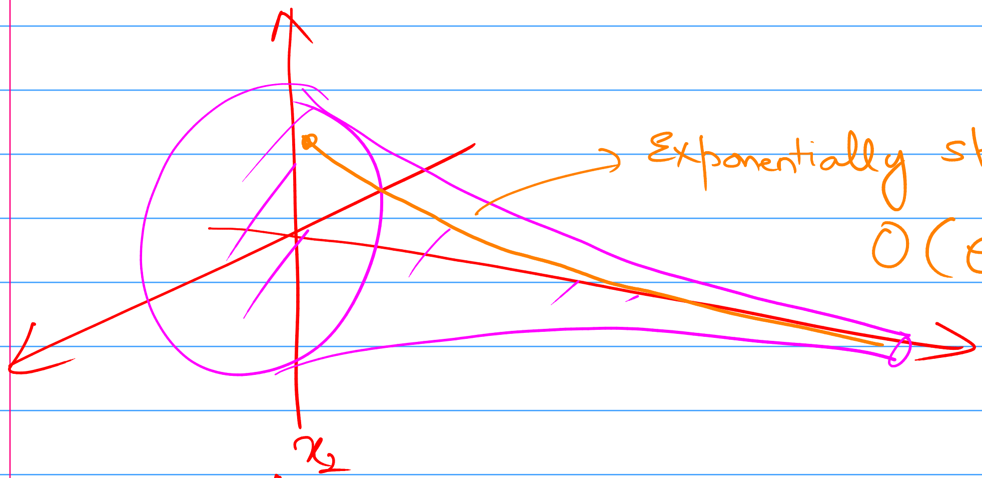
→ Stable in the sense of Lyapunov (Trajectories stay bounded for all $t \geq t_0$)



→ asymptotically stable

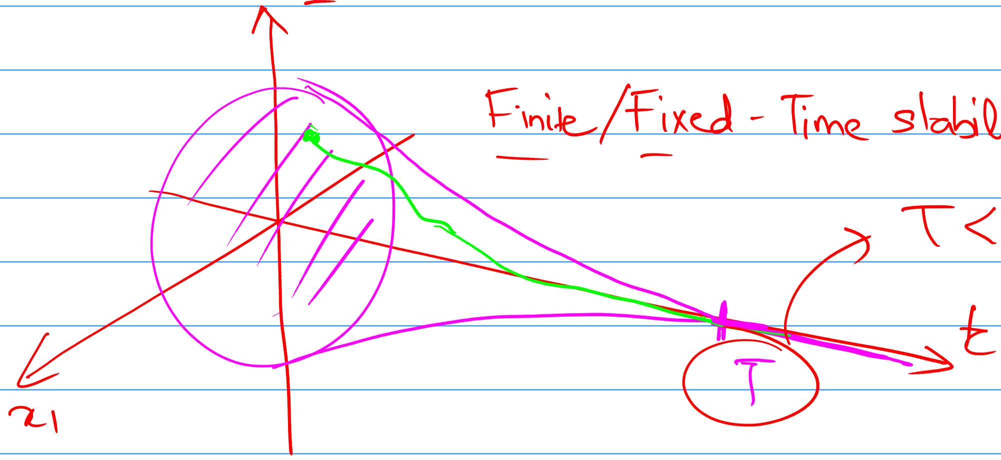
$O\left(\frac{1}{t}\right)$

$O\left(\frac{1}{t^2}\right)$



→ Exponentially stable

$O\left(e^{-\alpha t}\right)$



Finite/Fixed-Time stability

$T < \infty$

$\forall t \geq T$

$x(t) = 0$

$\dot{x} = g(x)$ s.t. $g(0) = 0$

* <u>Lyapunov stable</u>: Origin is stable in the sense of Lyapunov, if $\forall \varepsilon > 0 \ \exists \ \delta(t_0, \varepsilon)$ s.t.

$$\|x(t_0)\| < \delta \implies \|x(t)\| < \varepsilon \quad \forall t \geq t_0$$

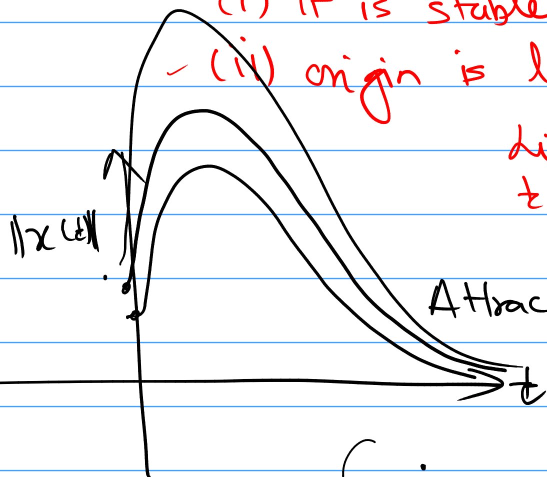
Uniformally stable if $\delta(\varepsilon)$.

* <u>Asymptotically Stable</u> (AS): Origin is A.S. if

(i) it is stable

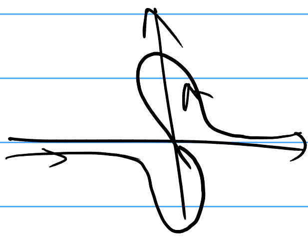
(ii) origin is locally attractive

$$\lim_{t \to \infty} x(t) = 0$$



Attractive but not stable.

Ex: Vinograd system



$$\begin{cases} \dot{x}_1 = \dfrac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)\left(1 + (x_1^2 + x_2^2)^2\right)} \\[2ex] \dot{x}_2 = \dfrac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)\left(1 + (x_1^2 + x_2^2)^2\right)} \end{cases}$$

* Exponentially <u>stable</u>: if $\|x(t)\| \leq m e^{-\alpha(t - t_0)} \quad \forall t \geq t_0$

$m, \alpha > 0$

$\hookrightarrow$ Lyapunov's direct method (using <u>Lyapunov functions</u>)

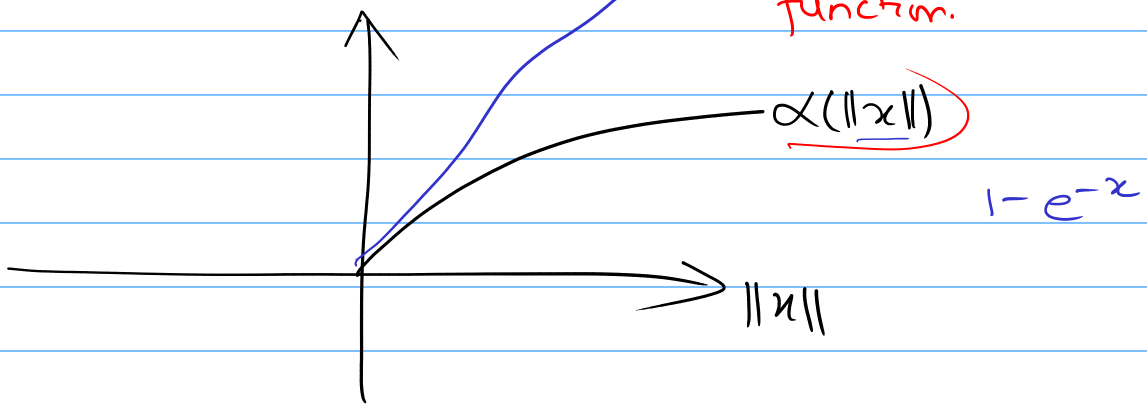
—x—x—x—x—x—x—

Lyapunov's Direct Method

Positive semi definite function: (PSD)

- $\underline{V(t,x)} \geq \alpha(\|x\|)$

$V(0)=0$ $\quad$ α is a strictly increasing function.



If $\|x\| \to \infty \Rightarrow \alpha(\|x\|) \to \infty$, then V is Positive Definite (Radial unboundedness)

$V(x) = \frac{1}{2}x^2$

V is negative definite if $-V$ is positive definite

* Stable $\quad$ If $V(x) > 0 \quad \forall x \setminus \{0\}$

$V(0)=0$

$\dot{V}(x) \leq 0$

* Asymptotically Stable: $\quad \left.\begin{array}{l} \text{If } V(x) > 0 \quad \forall x \setminus \{0\} \\ V(0)=0 \\ \dot{V} < 0 \quad \forall x \setminus \{0\} \end{array}\right\}$

Locally AS

* Exponentially Stable (ES):

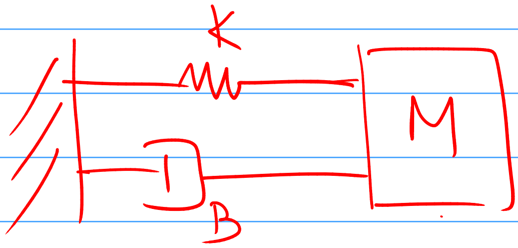
$$\begin{cases} V(x) > 0 & \forall x \setminus \{0\} \\ V(0) = 0 \\ \dot{V} \le -\alpha V, & \alpha > 0 \end{cases}$$

- does not imply that $x(t)$ converges exponentially fast.
- only guarantees exponential convergence of V.

$$\boxed{\begin{aligned} \alpha_1 \|x\|^2 \le V(x) &\le \alpha_2 \|x\|^2 \\ \dot{V} &\le -\alpha_3 \|x\|^2 \end{aligned}} \qquad \alpha_1, \alpha_2 > 0$$

$$\|x(t)\| \le m e^{-\alpha(t-t_0)} \qquad m \le \left(\frac{\alpha_2}{\alpha_1}\right)^{1/2} \qquad \alpha \ge \frac{\alpha_3}{2\alpha_2}$$

Ex:



$$\left.\begin{matrix} q \\ \dot{q} \end{matrix}\right\} \qquad V = \frac{1}{2} K q^2 + \frac{1}{2} M \dot{q}^2$$

$$M\ddot{q} + B\dot{q} + Kq = 0$$

$$\dot{x} = g(x)$$

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \qquad \underbrace{\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} \dot{q} \\ -\frac{K}{M} q - \frac{B}{M} \dot{q} \end{bmatrix}}_{g(x)}$$

$$g(x) = 0 \Rightarrow x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V = \frac{1}{2} \begin{bmatrix} q & \dot{q} \end{bmatrix} \underbrace{\begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}}_{} \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

$$V = \frac{1}{2}Kq^2 + \frac{1}{2}M\dot{q}^2$$

$$\begin{aligned}
\dot{V} &= Kq\dot{q} + \underline{M\ddot{q}\dot{q}} \\
&= Kq\dot{q} + \dot{q}(-B\dot{q} - Kq) \\
&= -B\dot{q}^2 \leq 0 \qquad \forall x \setminus \{x_e\}
\end{aligned}$$

$$\left.\begin{array}{ll} V > 0 & \forall x \setminus \{x_e\} \\ V(0) = 0 & \\ \dot{V} \leq 0 & \forall x \setminus \{x_e\} \end{array}\right] \longrightarrow \text{stability}$$

$$\left.\begin{array}{l} \text{stability} \\ \dot{V} \equiv 0 \end{array}\right\} \longrightarrow \text{A.S.}$$

Ex: $$V(x) = \frac{1}{2}\begin{bmatrix} q & \dot{q} \end{bmatrix} \underbrace{\begin{bmatrix} K & \epsilon M \\ \epsilon M & M \end{bmatrix}}_{A} \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

$\rightarrow$ V is PD.

$$\dot{V} = -\begin{bmatrix} q & \dot{q} \end{bmatrix} \underbrace{\begin{bmatrix} \epsilon K & \frac{1}{2}\epsilon B \\ \frac{1}{2}\epsilon B & B - \epsilon M \end{bmatrix}}_{\Delta} \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

$$\lambda_{min}\|x\|^2 \leq V(x) \leq \lambda_{max}(A)\|x\|^2 \longrightarrow \text{E.S.}$$

$$\dot{V} \leq -\lambda_{min}(\Delta)\|x\|^2$$

Exponentially Stable

Coming back to optimization:

Gradient Flows: $\dot{x} = -\nabla f(x) \longrightarrow f$ is μ-SC

$$V = \frac{1}{2}\|\nabla f\|^2 = \frac{1}{2}(\nabla f)^T(\nabla f)$$

$$\dot{V} = (\nabla f)^T(\nabla^2 f)\dot{x} = -(\nabla f)^T(\nabla^2 f)\nabla f$$

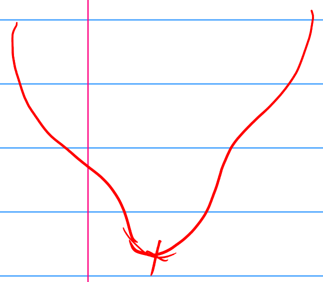
Since f is μ-SC $\Rightarrow \nabla^2 f \geq \mu I$

$$\dot{V} \leq -\mu\|\nabla f\|^2$$

$$\dot{V} \le -2\mu V$$

* Assume f satisfies PL-inequality.

$$\boxed{\frac{1}{2\mu}\|\nabla f\|^2 \ge (f(x) - f^*)}$$



$$V = f(x) - f^*$$

$$\dot{V} = \nabla f^T \dot{x}$$

$$= (\nabla f)^T(-\nabla f) = -\|\nabla f\|^2$$

$$= -\frac{1}{2\mu}(2\mu)\|\nabla f\|^2$$

$$\le -2\mu(f(x) - f_*)$$

$$\dot{V} \le -2\mu V$$