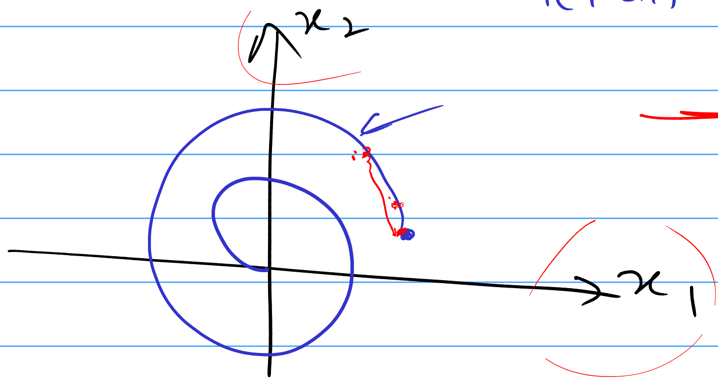
* RGF: $$\dot{x} = -\frac{\nabla f(x)}{\|\nabla f(x)\|^{\frac{p-2}{p-1}}}, \quad p > 2$$

→ Finite-time convergent

Lyapunov characterization:

$$\left.\begin{aligned} &\dot{V} \le -cV^{\alpha}, \quad \alpha \in (0,1) \\ &T(x_0) \le \frac{V(x_0)}{c(1-\alpha)} \end{aligned}\right\} \text{Finite-time stability}$$

Fixed-time stability: —
$$\left\{\begin{aligned} &\dot{V} \le -c_1 V^{\alpha_1} - c_2 V^{\alpha_2}, \quad \alpha_1 \in (0,1), \ \alpha_2 > 1 \\ &T \le \frac{1}{c_1(1-\alpha_1)} + \frac{1}{c_2(\alpha_2 - 1)} \end{aligned}\right.$$



$$\dot{x} = -\nabla f(x) \quad \text{— (1)}$$

$$\dot{x} = -\frac{\nabla f(x)}{\|\nabla f(x)\|^{\frac{p-2}{p-1}}}, \quad p > 2 \quad \text{— (2)}$$

$$\dot{x} = -\frac{\nabla f(x)}{\|\nabla f(x)\|^{\frac{p-2}{p-1}}} - \frac{\nabla f(x)}{\|\nabla f(x)\|^{\frac{q-2}{q-1}}}$$
$p > 2, \ q \in (1,2)$ — (3)

* If the function admits a quadratic growth,

$$x(k+1) = x(k) - \eta\left(\frac{\nabla f(x(k))}{\|\nabla f(x(k))\|^{\frac{p-2}{p-1}}} + \frac{\nabla f(x(k))}{\|\nabla f(x(k))\|^{\frac{q-2}{q-1}}}\right)$$

→ $\exists$ a small enough step-size s.t. independent of the IC, x would converge to an ε-ball around x^* in a fixed number of iterations.

* FxTS-GF (Fixed-Time Stable Gradient Flow)

Assume f is strictly convex; s.t $\nabla^2 f > 0$

When f was μ-SC:

$$\hookrightarrow \quad \dot{x} = -\frac{\nabla f(x)}{\|\nabla f(x)\|^{\frac{p-2}{p-1}}} - \frac{\nabla f(x)}{\|\nabla f(x)\|^{\frac{q-2}{q-1}}}, \quad p>2, \; q\in(1,2)$$

$$\dot{x} = -(\nabla^2 f)^{-1}\left(\frac{\nabla f}{\|\nabla f\|^{\frac{p-2}{p-1}}} + \frac{\nabla f}{\|\nabla f\|^{\frac{q-2}{q-1}}}\right), \quad p>2, \; q\in(1,2)$$

$$V = \frac{1}{2}\|\nabla f\|^2$$

$$\dot{V} = (\nabla f)^T(\nabla^2 f)\,\dot{x}$$

$$= -\frac{\|\nabla f\|^2}{\|\nabla f\|^{\frac{p-2}{p-1}}} - \frac{\|\nabla f\|^2}{\|\nabla f\|^{\frac{q-2}{q-1}}}$$

$$= -\|\nabla f\|^{2\cdot\frac{p}{2(p-1)}} - \|\nabla f\|^{2\cdot\frac{q}{2(q-1)}}$$

$$= -(2V)^{\frac{p}{2(p-1)}} - (2V)^{\frac{q}{2(q-1)}}$$

$$\dot{V} \leq -2^{\alpha_1}V^{\alpha_1} - 2^{\alpha_2}V^{\alpha_2}$$

$$\alpha_1 = \frac{p}{2(p-1)}$$

$$\alpha_2 = \frac{q}{2(q-1)}$$

$$\boxed{T \leq \frac{1}{2^{\alpha_1}(1-\alpha_1)} + \frac{1}{2^{\alpha_2}(\alpha_2-1)}}$$

$\alpha_1 \in (0,1)$

$\alpha_2 > 1$

$\uparrow$ Budget on total time required to solve the optimization problem

—— x —— x —— x —— x ——

* [Karimi 2016]

If a function f satisfies PL-inequality, then it admits at least a quadratic growth.

f satisfies PL-inequality with modulus $\mu > 0$

$$\left. f(x) - f^* \geq \frac{\mu}{2} \|x - x^*\|^2 \right\}$$

Proof: f satisfies PL-inequality

$$\frac{1}{2\mu} \|\nabla f(x)\|^2 \geq f(x) - f^* \quad \text{--- } (1)$$

$$g(x) := \sqrt{f(x) - f^*}$$

Since f satisfies PL-inequality
$\Downarrow$
f is an invex function
$\Downarrow$
g is a positive invex function

$$\nabla g(x) = \frac{\nabla f(x)}{2\sqrt{f(x) - f^*}}$$

$$\|\nabla g(x)\|^2 = \left(\frac{\|\nabla f(x)\|}{2\sqrt{f(x) - f^*}} \right)^2 = \frac{\|\nabla f(x)\|^2}{4(f(x) - f^*)} \geq \frac{\mu}{2}$$

$$\boxed{\|\nabla g(x)\|^2 \geq \frac{\mu}{2}} \quad \text{--- } (2)$$

Consider a gradient flow:

$$\dot{x} = -\nabla g(x)$$

$$g(x_0) - g(x(t)) = \int_{x(t)}^{x_0} \langle \nabla g(x), dx \rangle$$

$$= \int_t^0 \langle \nabla g(x), \frac{dx}{dt} \rangle dt$$

$$= -\int_0^t \langle \nabla g(x), \dot{x} \rangle dt$$

$$g(x_0) - g(x(t)) = \int_0^t \|\nabla g(x)\|^2 \, dt$$

$$\geq \frac{\mu t}{2}$$

$$g(x(t)) \leq g(x_0) - \frac{\mu t}{2}$$

Since $g(x(t))$ is positive, $\exists\, T < \infty$, s.t. $g(x(T)) = g(x^*)$

$$\mathcal{L}_{x_0}^{x^*} = \int_0^T \|\dot{x}\| \, dt$$

$$= \boxed{\int_0^T \|\nabla g(x)\| \, dt \geq \|x_0 - x^*\|} \quad \text{— } ③$$

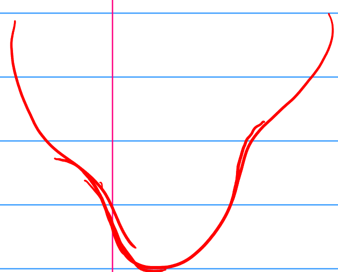
$$g(x_0) - g(x^*) = \int_0^T \|\nabla g(x)\|^2 \, dt$$

$$= \int_0^T \|\nabla g(x)\| \cdot \|\nabla g(x)\| \, dt$$

$$\geq \sqrt{\frac{\mu}{2}} \int_0^T \|\nabla g(x)\| \, dt \quad [\text{From } ②]$$

$$\geq \sqrt{\frac{\mu}{2}} \|x_0 - x^*\| \quad [\text{From } ③]$$

$x^2 + 3\sin^2 x$



$$g(x_0) \geq \sqrt{\frac{\mu}{2}} \|x_0 - x^*\| \quad \forall x_0$$

$$\boxed{f(x) - f^* \geq \frac{\mu}{2} \|x - x^*\|^2} \quad \text{— } \blacksquare$$

$*$ $g(x) := \underline{\|x\|^2} \longrightarrow$ strongly convex

$f(x) := \underline{\|Ax - b\|^2} \longrightarrow$ may or may not be SC

$\hookrightarrow$ SC only if A is full row-rank,

however, f satisfies PL-inequality $\forall A$

$\rightarrow$ [Karimi 2016]

Proof: Consider two points x and y.

Let $g(x)$ be σ-SC, we need to show that

$f(x) := g(Ax)$ satisfies PL-inequality for all A.

$u := Ax \qquad v := Ay \qquad \underline{\nabla f(x) = A^T \nabla g(Ax)}$

Since g is σ-SC,

$$g(v) \geq g(u) + \nabla g(u)^T (v - u) + \frac{\sigma}{2}\|v - u\|^2$$

$$f(y) \geq f(x) + \underbrace{\left(A^T \nabla g(Ax)\right)}_{\nabla f(x)}{}^T (y - x) + \frac{\sigma}{2}\|A(y - x)\|^2$$

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\sigma}{2}\|A(y - x)\|^2$$

$$f^* \geq f(x) + \nabla f(x)^T (x^* - x) + \frac{\sigma}{2}\|A(x^* - x)\|^2$$

— Hoffman, 1952

$$\geq \frac{\sigma\,\theta(A)}{2}\|x^* - x\|^2$$

where $\theta(A)$ is the smallest non-zero singular value of A.

$$\rightarrow f^* \geq f(x) + \nabla f(x)^T (x^* - x) + \frac{\sigma\,\theta(A)}{2}\|x^* - x\|^2$$

$$f^* \geq f(x) + \underbrace{\min_{y \in \mathbb{R}^n} \left[\nabla f(x)^T (y - x) + \frac{\sigma\,\theta(A)}{2}\|y - x\|^2\right]}$$

$$\nabla f(x) + \sigma\Theta(A)(y-x) = 0$$

$$(y-x) = -\frac{1}{\sigma\Theta(A)}\nabla f(x)$$

$$f^* \ge f(x) - \frac{1}{\sigma\Theta(A)}\|\nabla f(x)\|^2 + \frac{1}{2\sigma\Theta(A)}\|\nabla f(x)\|^2$$

$$f^* \ge f(x) - \frac{1}{2\sigma\Theta(A)}\|\nabla f(x)\|^2$$

$$\Rightarrow \frac{1}{2\sigma\Theta(A)}\|\nabla f(x)\|^2 \ge f(x) - f^*$$

$\Rightarrow$ f satisfies PL-inequality with $\mu = \sigma\Theta(A)$ —

★ Robustness of <u>FxTS-GF</u>:

$$\nabla f(x) \approx \frac{1}{n}\sum_{i=1}^{n}\nabla f(x_i)$$

$$\dot{x} = -c_1\frac{\nabla f(x)}{\|\nabla f(x)\|^{\frac{p-2}{p-1}}} - c_2\frac{\nabla f(x)}{\|\nabla f(x)\|^{\frac{q-2}{q-1}}} + \underline{\varepsilon(x)}$$

$$\underbrace{\|\varepsilon(x)\| \le l\,\|x-x^*\|^2}_{\text{Vanishing disturbance}}$$

If we choose c_1 and c_2 to be sufficiently large, then we would still converge in a fixed-amount of time.

<u>Assumption:</u> f satisfies PL-inequality with $\mu > 0$

$$\frac{1}{2\mu}\|\nabla f(x)\|^2 \ge f(x) - f^* \ge \frac{\mu}{2}\|x-x^*\|^2$$ — function has at least QG.

$$\ge \frac{\mu}{2l}\|\varepsilon(x)\|$$

$$\boxed{\|\varepsilon(x)\| \leq \underbrace{\frac{\ell}{\mu^2}}_{\bar{\ell}} \|\nabla f(x)\|^2} \quad \text{—} \ (1)$$

$$V = f(x) - f^*$$

$$\dot{V} = \nabla f^T \dot{x}$$

$$= \nabla f^T \left(-c_1 \frac{\nabla f}{\|\nabla f\|^{\frac{p-2}{p-1}}} - c_2 \frac{\nabla f}{\|\nabla f\|^{\frac{q-2}{q-1}}} + \varepsilon(x) \right)$$

$$= -c_1 \|\nabla f\|^{2 \cdot \frac{p}{2(p-1)}} - c_2 \|\nabla f\|^{2 \cdot \frac{q}{2(q-1)}} + \underbrace{\nabla f^T \varepsilon(x)}_{\leq \|\nabla f\| \|\varepsilon(x)\|}$$

CS

$$\boxed{\dot{V} \leq -c_1 \|\nabla f\|^{2 \cdot \frac{p}{2(p-1)}} - c_2 \|\nabla f\|^{2 \cdot \frac{q}{2(q-1)}} + \bar{\ell} \|\nabla f\|^3}$$

$$\boxed{\dot{V} \leq -c_1 \|\nabla f\|^{\frac{p}{p-1}} - c_2 \|\nabla f\|^{\frac{q}{q-1}} + \bar{\ell} \|\nabla f\|^3}$$

$p > 2$ $\quad 1 < \frac{p}{(p-1)} < 2$ $\qquad$ $q \in (1,2)$ $\quad \frac{q}{2(q-1)} > 1$

$\|\nabla f\| \leq 1$ $\quad \|\nabla f\|^2 \geq \|\nabla f\|^3$ $\qquad$ $\frac{q}{(q-1)} > 2 \rightarrow q < \frac{3}{2}$

$\|\nabla f\| \geq 1$ $\quad \|\nabla f\|^2 \leq \|\nabla f\|^3$ $\qquad$ $\frac{q}{q-1} > 3$

$$\dot{V} \leq -(c_1 - \bar{\ell}) \|\nabla f\|^{\frac{p}{p-1}} - (c_2 - \bar{\ell}) \|\nabla f\|^{\frac{q}{q-1}}$$

$$\dot{V} \leq -(c_1 - \bar{\ell}) \underbrace{\left(\|\nabla f\|^2\right)}_{1}^{\frac{p}{2(p-1)}} - (c_2 - \bar{\ell}) \|\nabla f\|^{2 \cdot \frac{q}{2(q-1)}}$$

$$\leq -(c_1 - \bar{\ell})(2\mu V)^{\frac{p}{2(p-1)}} - (c_2 - \bar{\ell})(2\mu V)^{\frac{q}{2(q-1)}}$$

We need $c_1, c_2 > \bar{\iota}$ and $q < \frac{3}{2} \implies \frac{q}{q-1} > 3$