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High energy physics, a levels and transitions approach

– Monograph –

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Dedicated to Vijay Laxmi Chootni
Preface

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Part I
Chapter 1
Introduction
Consider lab frame $O$ and a frame $O'$, moving with respect to lab frame with velocity $v$ in the x-direction as shown in Fig. 2.1. Then the space time increment $(\Delta x, \Delta t)$ in $O$, corresponds to $(\Delta x', \Delta t')$ in $O'$. The phase increment of the light wave in both frames is the same. The velocity of light is same in both frames, which is the central tenet of theory of relativity.

Then

\[ k \Delta x - \omega \Delta t = k' \Delta x' - \omega' \Delta t' \]  \hspace{1cm} (2.1)

\[ k(\Delta x - c\Delta t) = k'(\Delta x' - c\Delta t'). \]  \hspace{1cm} (2.2)

For light travelling in opposite direction

\[ k'(\Delta x + c\Delta t) = k(\Delta x' + c\Delta t'). \]  \hspace{1cm} (2.3)

The two relations give
\[ (c\Delta t)^2 - \Delta x^2 = (c\Delta t')^2 - \Delta x'^2. \] (2.4)

For \( \Delta x' = 0 \), we have, \( \Delta x = v\Delta t \) and this gives

\[ \Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} \] (2.5)

This is called time dilation. Furthermore

\[ k' = \frac{1 - \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} \] (2.6)

Then combining Eq. (2.2, 2.3, 2.6), we get

\[ \begin{bmatrix} \frac{\Delta x}{c\Delta t} \\ \frac{\Delta x'}{c\Delta t'} \end{bmatrix} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \begin{bmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{bmatrix} \begin{bmatrix} \frac{\Delta x'}{c\Delta t'} \end{bmatrix} \] (2.7)

For a rod of length \( l' \) in \( O' \) we have \( (\Delta x', \Delta t') = (l', 0) \), the \( l = \Delta x - v\Delta t = l'\sqrt{1 - \frac{v^2}{c^2}} \). This is called length contraction.

For an object moving at velocity in the frame \( O' \) at velocity \( u \), for time \( \Delta t' \), we have \( (\Delta x', \Delta t') = (u\Delta t', \Delta t') \). Then from (Eq. 2.7), the relative velocity

\[ v = \frac{\Delta x}{\Delta t} = \frac{u + \frac{v}{1 + \frac{v^2}{c^2}}} \] (2.8)

Consider a electron matter wave with frequency, wavevector \((\omega, k)\) and \((\omega', k')\) respectively. Then

The phase increment of the matter wave in both frames is the same.

Then

\[ k\Delta x - \omega\Delta t = k'\Delta x' - \omega'\Delta t' \] (2.9)

\[ \begin{bmatrix} k - \frac{\omega}{c} \\ \frac{\Delta x}{c\Delta t} \end{bmatrix} = \begin{bmatrix} k - \frac{\omega}{c} \end{bmatrix} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \begin{bmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{bmatrix} \begin{bmatrix} \Delta x' \\ c\Delta t' \end{bmatrix} = \begin{bmatrix} k' - \frac{\omega'}{c} \end{bmatrix} \begin{bmatrix} \Delta x' \\ c\Delta t' \end{bmatrix} \] (2.10)

This gives

\[ \begin{bmatrix} k - \frac{\omega}{c} \end{bmatrix} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \begin{bmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{bmatrix} = \begin{bmatrix} k' - \frac{\omega'}{c} \end{bmatrix} \] (2.11)

Rewriting this equation we get
Then the energy
\[ E(v) = \hbar \omega(v) = \frac{\hbar}{\sqrt{1 - \frac{v^2}{c^2}}} (\nu k' + \omega'). \] (2.13)

Consider a classical particle of mass \( m \) moving with velocity \( v \) in frame \( O' \). Its kinetic energy is \( \frac{1}{2} mv^2 \). If no work is done on the system with a force then this energy is conserved. Furthermore this is conserved in all frames of reference. The kinetic energy in frame \( O \) in which the frame \( O' \) moves with velocity \( u \) is
\[ E(u) = \frac{1}{2} m (v + u)^2. \]
For infinitesimal \( u \), \( E(u) \sim \frac{1}{2} mv^2 + mvu \) and since energy is conserved in this new frame \( mv = \frac{dE}{du} \) is called momentum. Using this interpretation of momentum, we can calculate the momentum of the complex wave.

Once again we use our interpretation of momentum and ask what is \( \frac{dE(v)}{dv} \big|_0 = \hbar k' \). Therefore momentum of our complex wave \( \omega', k' \) is simply
\[ \hbar k'. \]

Thus we have two basic results in quantum mechanics the energy is \( \hbar \omega \) and momentum \( \hbar k \).

Consider a matter wave with energy, momentum \( (E, p) \) and \( (E', p') \) respectively. Then

The phase increment of the matter wave in both frames is the same.

Then
\[ p \Delta x - E \Delta t = p' \Delta x' - E' \Delta t' \] (2.14)

This gives
\[ [p - \frac{E}{c}] \left[ \frac{\Delta x}{c \Delta t} \right] = [p - \frac{E}{c}] \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left[ \frac{\Delta x'}{c \Delta t'} \right] = [p' - \frac{E'}{c}] \frac{\Delta x'}{c \Delta t'} \] (2.15)

Combining Eq. (2.15 and 2.16), we get
\[ E^2 - (pc)^2 = E'^2 - (p'c)^2 \] (2.17)

Let the energy of the mass \( m_0 \) in \( O' \) be \( E'_0 \). Then its energy is \( O \) is \( E'^2_0 + (mvc)^2 \), let this mass disintegrate giving two photons in forward and backward direction of
energy $\hbar \omega_0$ each. Then in frame 0, the energies of photons are $\hbar \omega_1$ and $\hbar \omega_2$. Then we get

$$\hbar \omega_1 + \hbar \omega_2 = \sqrt{(2\hbar \alpha_0)^2 + (mc)^2}$$

This gives using $\frac{1}{2}(\frac{\partial \beta_0}{\partial \beta_0} + \frac{\partial \beta_0}{\partial \beta_0}) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$,

$$1 + \frac{(mc)^2}{E_0^2} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This gives $E_0^2 = (mc^2)^2(1 - \frac{v^2}{c^2})$ and $E^2_0 = (mc^2)^2$. This gives $E_0' = m_0 c^2$ and $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$.

### 2.1 Photon

Consider Maxwell equations in free space in coordinate system $O$.

\[ \nabla \cdot E = 0, \quad \text{(2.18)} \]
\[ \nabla \cdot B = 0, \quad \text{(2.19)} \]
\[ \nabla \times B = \mu_0 \frac{\partial D}{\partial t}, \quad \text{(2.20)} \]
\[ \nabla \times E = -\frac{\partial B}{\partial t}. \quad \text{(2.21)} \]

In coordinate system $O'$, the $E, B$ transform to $E', B'$ such that Maxwell equations stay same, i.e., the new $E'$ and $B'$ fields should also satisfy Maxwell equations. So what should be the transformation rule. Recall, we call write from 2.19 that

$$B = \nabla \times A. \quad \text{(2.22)}$$

$(A = (A_x, A_y, A_z))$ and substituting in 2.21, we get

$$E = -\frac{\partial A}{\partial t} - \nabla A_0. \quad \text{(2.23)}$$

With $A^\mu = (A_0, A)$, observe the gauge Transformation

$$A_{\mu} \rightarrow A_{\mu} - \partial_{\mu} \chi, \quad \text{(2.24)}$$

does not change $E$ and $B$ so we choose Lorentz gauge

$$\partial_{\mu} A^\mu = 0. \quad \text{(2.25)}$$
Substituting for $E, B$ in terms of $A$ in Eq. 2.18 and 2.20, we find

$$\partial^\mu \partial^\nu A^\nu = 0.$$  \hspace{1cm} (2.26)

Now define

$$A'(x'(x)) = \Lambda A(x).$$  \hspace{1cm} (2.27)

Then we can check that

$$\partial_\mu A'^\mu = 0,$$  \hspace{1cm} (2.28)

and

$$\partial_\mu \partial_\nu A'^\nu = 0.$$  \hspace{1cm} (2.29)

Now we can define $E'$ and $B'$ in terms of $A'$ as in 2.22 and 2.23, this insures that $B'$ and $E'$ satisfy Maxwell equation 2.19 and 2.21. Then using 2.28 and 2.29, we get $E'$ and $B'$ also satisfy 2.18 and 2.20. Therefore the new $E'$ and $B'$ fields also satisfy Maxwell equations. Remember, transformation rule is 2.27.

Electromagnetic field is

$$F^{\mu \nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}.$$  \hspace{1cm} (2.30)

We know give a variational interpretation to the equation of $A$ field 2.29. We equip $A$ with a dynamics by defining Lagrangian as density

$$L = -\frac{\varepsilon_0}{4} F_{\mu \nu} F^{\mu \nu} = \frac{\varepsilon_0}{2} (E^2 - B^2).$$  \hspace{1cm} (2.31)

The corresponding energy density is

$$H = \varepsilon_0 (F_{0 \mu} F^{0 \mu} + \frac{1}{4} F_{\mu \nu} F^{\mu \nu}) = \frac{\varepsilon_0}{2} (E^2 + B^2).$$  \hspace{1cm} (2.32)

Once we have the Lagrangian, we can write the Euler Lagrange equations that give us Eq. 2.29. Let's make a small detour on how Euler Lagrange equations arise from Lagrangian.

### 2.1.1 Euler Lagrange Equations

Recall given a mechanical system with Lagrangian $L(x, \dot{x})$, we want to find the trajectory connecting two fixed points that minimize

$$S = \int L(x, \dot{x}) \, dt.$$  \hspace{1cm} (2.33)
\[ \delta S = \int \delta L(x, \dot{x}) \, dt, \] (2.34)

\[ \delta S = \int \left( \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right), \] (2.35)

Integrating by parts with \( \delta x \) as 0 at the endpoints/boundary, we have

\[ \delta S = \int \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \delta x. \] (2.36)

For above to be true for arbitrary \( \delta \) we have

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}. \] (2.37)

As an example consider a spring mass system with mass \( m \) and spring constant \( k \), then

\[ L(x, \dot{x}) = \frac{1}{2} (m \dot{x}^2 - k x^2). \] (2.38)

Then Euler Lagrange equations read

\[ m \ddot{x} = -k x. \] (2.39)

### 2.1.2 Klein Gordon Field

Consider a scalar field with Lagrangian density

\[ L = \frac{1}{2} \left( \partial_{\mu} \phi \partial^{\mu} \phi - m^2 \phi^2 \right). \] (2.40)

Given the action

\[ S = \int L \, d^3 x \] (2.41)

Then

\[ \delta S = \int \partial_{\mu} \phi \partial^{\mu} \delta \phi - m \phi \delta \phi \, d^3 x \] (2.42)

\[ \delta S = \int \left( -\partial_{\mu} \partial^{\mu} \phi - m^2 \phi \right) \delta \phi \, d^3 x \] (2.43)

where we integrate by parts with variation zero at boundary and we get for arbitrary \( \delta \phi \), it should be true that

\[ \partial_{\mu} \partial^{\mu} \phi + m^2 \phi = 0, \] (2.44)

or
\[
\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + m^2 \phi = 0. \quad (2.45)
\]

### 2.1.3 Variation of \( A \) field

We equip \( A \) with a dynamics by defining Lagrangian as density

\[
L = -\frac{\varepsilon_0}{4} F_{\mu \nu} F^{\mu \nu}. \quad (2.46)
\]

Then

\[
S = \int L \, d^3 x
\]

(2.47)

Then

\[
\delta S = \int F_{\mu \nu} (\partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu) \, d^3 x, \quad \text{(2.48)}
\]

\[
\delta S = \int \partial^\mu F_{\mu \nu} \delta A^\nu \, d^3 x, \quad \text{(2.49)}
\]

where we used integration by parts and zero variation at the boundary to get

\[
\partial^\mu F_{\mu \nu} = \partial^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu). \quad \text{(2.50)}
\]

Invoking the Lorentz gauge we get

\[
\partial^\mu \partial_\mu A_\nu = 0, \quad \text{(2.51)}
\]

i.e.,

\[
\left( \frac{c^2}{\partial t^2} - \nabla^2 \right) A_\nu = 0. \quad \text{(2.52)}
\]

The solution is for \( \varepsilon^\mu = (\varepsilon^0, \varepsilon) \), we have

\[
A = \varepsilon \cos(k \cdot r - \omega t), \quad \text{(2.53)}
\]

is a wave propagating in \( k = (k_x, k_y, k_z) \) direction with \( \omega = c |k| \). with \( k^\mu = (\frac{\omega}{c}, \mathbf{k}) \).

The Lorentz gauge condition then becomes

\[
k_\mu \varepsilon^\mu = 0. \quad \text{(2.54)}
\]

For example,

\[
A = \varepsilon \cos(k \cdot z - \omega t), \quad \text{(2.55)}
\]

is a wave propagating in \( z \) direction with \( \omega = c |k| \). From 2.32, the energy of this \( A \) field is \( (\varepsilon_z^2 + \varepsilon^2) \frac{\partial \omega}{c^2} V \). Therefore for \( \varepsilon_z^2 + \varepsilon^2 = 1 \), we have,
\[ A = c \sqrt{\frac{2\hbar}{\epsilon_0 \omega V}} \cos(k \cdot z - \omega t) = c \sqrt{\frac{\hbar}{2\epsilon_0 \omega V}} \epsilon \left(\exp i(k \cdot z - \omega t) + \exp -i(k \cdot z - \omega t)\right) \]  

(2.56)

has energy \( \hbar \omega \), and this elementary excitation is termed \textbf{Photon}. More generally, when \( \epsilon \) is complex

\[ A = c \sqrt{\frac{\hbar}{2\epsilon_0 \omega V}} \left( \epsilon \exp i(k \cdot z - \omega t) + \epsilon^* \exp -i(k \cdot z - \omega t) \right) \]  

(2.57)

For instance if \( \epsilon = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix} \), we have

\[ A = c \sqrt{\frac{\hbar}{\epsilon_0 \omega V}} \begin{bmatrix} 0 \\ \cos(k \cdot z - \omega t) \\ \sin(k \cdot z - \omega t) \\ 0 \end{bmatrix} \]  

(2.58)

constitutes circularly polarized light. If we move into a frame that rotates around \( z \) axis with angular velocity \( \Delta \omega \), the find the \( A \) transforms to

\[ A' = c \sqrt{\frac{\hbar}{\epsilon_0 \omega V}} \begin{bmatrix} 0 \\ \cos(k \cdot z - (\omega + \Delta \omega) t) \\ \sin(k \cdot z - (\omega + \Delta \omega) t) \\ 0 \end{bmatrix} \]  

(2.59)

If we calculate the energy of \( A' \), we get two contributions, one due to \( z \) dependence of \( \frac{\hbar \omega}{2} \) and \( t \) dependence, which is \( \frac{\hbar(\omega + \Delta \omega)^2}{2 \omega} \). Going from \( A \) to \( A' \) the energy changes by \( \Delta E = \hbar \Delta \omega \) and the angular momentum is just \( \frac{\Delta E}{\Delta \omega} = \hbar \). Thus circularly polarized light carries angular momentum of \( \hbar \).

### 2.2 Electron

We now come to relativistic equation of an electron. Recall Pauli matrices

\[ \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \ \sigma_y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}; \ \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \ \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]  

(2.60)

\[ i\hbar \frac{\partial \psi}{\partial t} = (mc^2 \beta - i\hbar \alpha_j \partial_j) \psi. \]
where $\beta = \sigma_z \otimes 1$ and $\alpha = -\sigma_z \otimes \sigma_i$. Using $\hbar = c = 1$ and multiplying both sides with $\beta$ gives

$$(i\gamma^\mu \partial_\mu - m)\psi = 0,$$

where $\gamma^0 = \sigma_z \otimes 1$ and $\gamma^i = i\sigma_y \otimes \sigma_i$. Let us diagonalize the matrix

$$C = m\beta + p_j \alpha_j = E(-\cos \theta \sigma_z \otimes \sigma_\alpha + \sin \theta \sigma_i \otimes 1) \quad (2.61)$$

$$= E \exp(i\frac{\theta}{2} \sigma_y \otimes \sigma_\alpha) (-\sigma_z \otimes \sigma_\alpha) \exp(-i\frac{\theta}{2} \sigma_y \otimes \sigma_\alpha) \quad (2.62)$$

Let $\xi_{\pm}$ be eigenvectors of $\sigma_\alpha$ with eigenvalues $\pm 1$. Then $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \xi_{\pm}$ are eigenvectors of $-\sigma_z \otimes \sigma_\alpha$ with eigenvalues $\pm 1$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \xi_{\pm}$ are eigenvectors of with eigenvalues $\mp 1$. Then

$$\exp(i\frac{\theta}{2} \sigma_y \otimes \sigma_\alpha) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \xi_{\pm} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \pm \sin \frac{\theta}{2} \end{bmatrix} \otimes \xi_{\pm} \quad (2.63)$$

are eigenvectors of $C$ with eigenvalues $\pm E$.

$$\exp(i\frac{\theta}{2} \sigma_y \otimes \sigma_\alpha) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \xi_{\pm} = \begin{bmatrix} \mp \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix} \otimes \xi_{\pm} \quad (2.64)$$

are eigenvectors of $C$ with eigenvalues $\mp E$. let $u$ and $v$ be these eigenvectors with $\pm$ eigenvalues respectively. Then let

$$u_1(p) = \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} \otimes \xi_{\alpha}^+; \quad u_2(p) = \begin{bmatrix} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix} \otimes \xi_{\alpha}^-; \quad (2.65)$$

$$v_1(p) = \begin{bmatrix} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix} \otimes \xi_{\alpha}^+; \quad v_2(p) = \begin{bmatrix} \cos \frac{\theta}{2} \\ -\sin \frac{\theta}{2} \end{bmatrix} \otimes \xi_{\alpha}^- \quad (2.66)$$

where 1,2 represent positive and negative helicity respectively.

### 2.2.1 Completeness Relation

Then using

$$\sum_{s=1,2} u_s(p)u_s(p)^\dagger + v_s(p)v_s(p)^\dagger = 1 \quad (2.67)$$

and

$$E_p \sum_{s=1,2} u_s(p)u_s(p)^\dagger - v_s(p)v_s(p)^\dagger = (mc^2 \beta + c \alpha, p_i) \quad (2.68)$$
This gives for \( m = mc^2 \),

\[
\sum_{s=1,2} u_s(p)\bar{u}_s(p) = \frac{\not{p} + m}{2E_p} \quad (2.69)
\]

\[
\sum_{s=1,2} v_s(p)\bar{v}_s(p) = \frac{\not{p} - m}{2E_p}. \quad (2.70)
\]

where \( \not{p} = p\gamma^\mu \).

Coming back to Dirac equation

\[
(\gamma^\mu p_\mu + m)u(p) = 0. \quad (2.71)
\]

Let momentum \( p' \) is related to \( p \) by a Lorentz transformation. The Lorentz transformation that takes \( p \) to \( p' \) can be written as boost from \( p \) to rest and arbitrary rotation and then boost from rest to \( p' \). We can represent this as

\[
\Lambda = B_{p'}(\xi')\exp(\theta_1\Omega_\alpha)B_p(-\xi), \quad (2.72)
\]

where \( B_x \) is boost along \( x \) direction. On spinor, it takes the form,

\[
u(p') = \Sigma u(p), \quad (2.73)
\]

where

\[
\Sigma = \exp(-\frac{\xi'}{2}\sigma_z \otimes \sigma_p)\exp(i\frac{\theta_1}{2}\sigma_\alpha \otimes \sigma_p)\exp\left(\frac{\xi}{2}\sigma_z \otimes \sigma_p\right). \quad (2.74)
\]

Then it can be verified that

\[
(\gamma^\mu p_\mu' + m)u(p') = 0. \quad (2.75)
\]

It should be noted that Eq. (2.73) does not preserve norm. However what is true is that if we normalize the spinors such that \( u(p) = \sqrt{E} u(p) \) then under Lorentz transformation,

\[
\Sigma u(p) = u(p'). \quad (2.76)
\]

To see this, let

\[
u(p) = \begin{bmatrix} \cos \frac{\theta(p)}{2} \\ \sin \frac{\theta(p)}{2} \end{bmatrix} \otimes \xi_p. \quad (2.77)
\]

Consider a spinor \( u(0) = \sqrt{\frac{E}{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \xi_p \) at rest. If we boost it to momentum \( p \), we can write \( u(0) \rightarrow w \).

\[
w = \exp(-\frac{\xi}{2}\sigma_z \otimes \sigma_p)u(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} \exp(-\frac{\xi}{2}) \\ \exp(\frac{\xi}{2}) \end{bmatrix} \otimes \xi_p. \quad (2.78)
\]
Since \( w \propto u(p) \), we have
\[
\tan \frac{\theta(p)}{2} = \exp(\zeta). \tag{2.79}
\]
Then
\[
|w|^2 = m \cosh(\zeta) = \frac{m}{\sin(\theta(p))} = E_p, \tag{2.80}
\]
but this says that
\[
u(p) = w. \tag{2.81}
\]
Then from 2.74,
\[
\Sigma \nu(p) = \exp(-\frac{\zeta'}{2} \sigma_z \otimes \sigma_{p'}) \exp(i \frac{\theta(p)}{2} \sigma_0) \nu(0), \tag{2.82}
\]
\[
= \sqrt{m} \exp(-\frac{\zeta'}{2} \sigma_z \otimes \sigma_{p'}) (a \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \xi_{p'}^+ + b \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \xi_{p'}^-), \tag{2.83}
\]
where \( a^2 + b^2 = 1 \). Then
\[
\Sigma \nu(p) = \sqrt{m} (a \frac{1}{\sqrt{2}} \begin{bmatrix} \exp(-\frac{\zeta'}{2}) \\ \exp(\frac{\zeta'}{2}) \end{bmatrix} \otimes \xi_{p'}^+ + b \frac{1}{\sqrt{2}} \begin{bmatrix} \exp(\frac{\zeta'}{2}) \\ \exp(-\frac{\zeta'}{2}) \end{bmatrix} \otimes \xi_{p'}^-), \tag{2.84}
\]
\[
\propto (a \begin{bmatrix} \cos \frac{\theta(p')}{2} \\ \sin \frac{\theta(p')}{2} \end{bmatrix} \otimes \xi_{p'}^+ + b \begin{bmatrix} \sin \frac{\theta(p')}{2} \\ \cos \frac{\theta(p')}{2} \end{bmatrix} \otimes \xi_{p'}^-). \tag{2.85}
\]
Then equating coefficients of \( \xi_{p'} \) we get \( \propto \) in last equation is \( \sqrt{E_{p'}} \) and
\[
\Sigma \nu(p) = \nu(p'). \tag{2.86}
\]

### 2.3 Electron-Photon Interaction

#### 2.3.1 Electric-Magnetic Field Lagrangian and Hamiltonian

A charged particle with mass \( m \) and charge \( q \) in electric-magnetic field has Lagrangian
\[
L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + q(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) - qA_0, \tag{2.87}
\]
where \( \mathbf{A} \) is vector potential and \( \mathbf{B} = \nabla \times \mathbf{A} \), i.e.,
\[ B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \quad (2.88) \]
\[ B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \quad (2.89) \]
\[ B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}. \quad (2.90) \]

\[ E_x = -\frac{\partial A_x}{\partial t} - \frac{\partial A_0}{\partial x}, \quad (2.91) \]
\[ E_y = -\frac{\partial A_y}{\partial t} - \frac{\partial A_0}{\partial y}, \quad (2.92) \]
\[ E_z = -\frac{\partial A_z}{\partial t} - \frac{\partial A_0}{\partial z}. \quad (2.93) \]

We have Euler Lagrange equations

\[ m\ddot{x} + q\dot{A}_x = q(x\frac{\partial A_x}{\partial x} + y\frac{\partial A_y}{\partial x} + z\frac{\partial A_z}{\partial x}) - q\frac{\partial A_0}{\partial x}. \quad (2.94) \]

Writing

\[ \dot{A}_x = \frac{\partial A_x}{\partial t} + \dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}, \quad (2.95) \]

Substituting in 2.94 we get,

\[ m\ddot{x} = q(\dot{y}B_z - \dot{z}B_y) - q(\frac{\partial A_x}{\partial t} + \frac{\partial A_0}{\partial x}), \quad (2.96) \]

and similarly for \( y, z \) gives in all that

\[ m\ddot{v} = q(\nu \times B + E). \quad (2.97) \]

The Loretz force law. The momentum \( p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x \) and similarly for \( y, z \) gives the Hamiltonian or energy

\[ H = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - L = \frac{(p_x - qA_x)^2}{2m} + \frac{(p_y - qA_y)^2}{2m} + \frac{(p_z - qA_z)^2}{2m} + qA_0. \quad (2.98) \]

The energy \( E \) is then

\[ E - qA_0 = \frac{(p_x - qA_x)^2}{2m} + \frac{(p_y - qA_y)^2}{2m} + \frac{(p_z - qA_z)^2}{2m}. \quad (2.99) \]

The relstisic generalization is

\[ E - qA_0 = c\sqrt{(p_x - qA_x)^2 + (p_y - qA_y)^2 + (p_z - qA_z)^2 + (mc)^2}. \quad (2.100) \]
We use this energy to define Dirac equation in the electric-magnetic field.

### 2.3.2 Gauge Coupling and Transitions

Recall how the Dirac equation reads with \( p^\mu = (E, p_c) \), we have

\[
(\gamma^\mu p_\mu - m)\psi = 0 \tag{2.101}
\]

From 2.100, we have the Dirac equation in presence of electromagnetic field as \( p_\mu \to p_\mu - qA_\mu \). If we identify \( p_\mu \) with \( i\partial_\mu \), the Dirac equation reads

\[
(i\gamma^\mu \partial_\mu - mc - qA_\mu \gamma^\mu)\psi = 0 \tag{2.102}
\]

or in terms of matrices \( \alpha_j, \beta \),

\[
i\partial_\mu \psi = (-c\alpha_j i\partial_j + mc^2\beta - qA_j\alpha_j + qA_0)\psi \tag{2.103}
\]

without \( A \), we have the free Dirac equation

\[
i\partial_\mu \psi = (-c\alpha_j i\partial_j + mc^2\beta)\psi \tag{2.104}
\]

The term

\[
T = -qA_j\alpha_j + qA_0, \tag{2.105}
\]

is transition term. In absence of this if electron is in eigenstate of the Dirac equation \( \psi_1 = u_1(p)\exp(ip\cdot r) \), it will stay in this state. In presence of the transition term it makes a transition. Lets imagine a photon is present, then \( A \) is as in Eq. 2.106

\[
A = c\sqrt{\frac{2\hbar}{\varepsilon_0\omega V}}\cos(k\cdot r - \omega t) = c\sqrt{\frac{\hbar}{2\varepsilon_0\omega}}(\exp i(k\cdot r - \omega t) + \exp -i(k\cdot r - \omega t)) \tag{2.106}
\]

Then \( T \) acting on \( \psi_1 \) induces change \( u_1(p)\exp(ip\cdot r) \to u_1(p+k)\exp(i(p+k)\cdot r) \). Lets call the state \( u_1(p+k)\exp(i(p+k)\cdot r) \) as \( \psi_2 \). Let \( x_1 \) and \( x_2 \) denote coefficients of \( \psi_1 \) and \( \psi_2 \). Then

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{-i}{\hbar} \begin{bmatrix} E_p & \Omega^\dagger \exp(i\omega t) \\ \Omega \exp(-i\omega t) & E_{p+k} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.107}
\]

where \( E_p \) and \( E_{p+k} \) are energies of the initial and final electron states and
\[
\Omega = q \sqrt{\frac{\hbar}{2VE_0\omega}} \ u_1^\dagger(p+k)(-\varepsilon_j\alpha_j + \varepsilon_0)u_1(p) = \frac{C}{\sqrt{2E_k}} u_1(p+k)(\varepsilon_\mu\gamma^\mu)u_1(p),
\]
where \( \bar{u} = u^\dagger \beta \), \( E_k = \hbar\omega \) the energy of the photon and
\[
C = \frac{qc\hbar}{\sqrt{V\varepsilon_0}}.
\]
If we denote \( \tilde{x}_1 = \exp(-i\omega t)x_1 \), then
\[
\frac{d}{dt} \begin{bmatrix} \tilde{x}_1 \\ x_2 \end{bmatrix} = -i \frac{\hbar}{\hbar} \begin{bmatrix} E_p + E_k & \Omega^* \\ \Omega & E_{p+k} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ x_2 \end{bmatrix}
\]
where \( E_k = \hbar\omega_k \) the energy of the photon. Observe \( \tilde{x}_1 \) is coefficient of state evolving with energy \( E_p + E_k \), the joint state of electron and photon. \( d \) then represent the transition out of this state. The initial state has electron and photon, while the final state only has electron. We denote this as
\[
|p,k\rangle \rightarrow |p+k\rangle
\]
Drawn as an energy level diagram it looks like Fig. 2.2.

![Energy Level Diagram](image)

**Fig. 2.2** Fig. shows transitions between electron, photon states.

We showed transition to state \( u_1(p+k) \), similarly we have transition to state \( u_2(p+k) \), etc, where 1, 2 represent helicity.
2.3 Electron-Photon Interaction

2.3.3 Feynman Diagrams

The above level diagram is also represented by so called a Feynman diagram as shown in 2.3. An electron and photon of momentum \( p \) and \( k \) respectively react to form an electron of momentum \( p+k \).

![Feynman diagram](image)

Fig. 2.3 Fig. shows a Feynmann diagram of electron and photon of momentum \( p \) and \( k \) respectively react to form an electron of momentum \( p+k \)

This reaction is very unfavourable for large energy difference between input and out states of the two levels as shown in fig. 2.2. However the outgoing electron can immediately dissociate into an electron and photon of momentum \( p' \) and \( k' \) such that \( p'+k'=p+k \). Furthermore \( E_p + E_k = E_{p'} + E_{k'} \) so that the initial and final states have same energy-momentum. Then the overall reaction which is represented by a level diagram in 2.4A and Feynman diagram 2.4B becomes favorbale. We can compute the rate of this reaction, which we do in the next chapter on Quantum Electrodynamics a subject that calculates such reaction rates.
Fig. 2.4 Fig. A shows a three level diagram and Fig. B shows a Feynman diagram for electron-photon scattering.
Chapter 3
Quantum Electrodynamics

3.1 Introduction

We first develop an analogy between the three level atomic system so called $\Lambda$ system and scattering processes in quantum electrodynamics (QED) \cite{8, 9, 10, 11}. In a $\Lambda$ system as shown in Fig. 3.1 we have two ground state levels $|1\rangle$ and $|3\rangle$ at energy $E_1$ and excited level $|2\rangle$ at energy $E_2$. The transition from $|1\rangle$ to $|2\rangle$ has strength $\Omega_1$ and transition from $|2\rangle$ to $|3\rangle$ has strength $\Omega_2$. In the interaction frame of natural Hamiltonian of the system, we get a second order term connecting level $|1\rangle$ to $|3\rangle$ with strength $\frac{\Omega_1 \Omega_2}{(E_1 - E_2)}$. This term creates an effective coupling between ground state levels and drives transition from $|1\rangle$ to $|3\rangle$. Scattering processes in QED can be modelled like this. Feynman amplitudes are calculation of second order term $M = \frac{\Omega_1 \Omega_2}{(E_1 - E_2)}$.

Fig. 3.1 Above Fig. shows a three level $\Lambda$ system with two ground state levels $|1\rangle$ and $|3\rangle$ and an excited level $|2\rangle$. 
The state of the three level system evolves according to the Schrödinger equation
\[ \dot{\psi} = -\frac{i}{\hbar} \begin{bmatrix} E_1 & \Omega_1^* & 0 \\ \Omega_1 & E_2 & \Omega_2^* \\ 0 & 0 & E_1 \end{bmatrix} \psi. \] (3.1)

We proceed into the interaction frame of the natural Hamiltonian (system energies) by transformation
\[ \phi = \exp\left(\frac{i}{\hbar} \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_1 \end{bmatrix} \right) \psi. \] (3.2)

This gives for \( \Delta E = E_2 - E_1 \),
\[ \dot{\phi} = -\frac{i}{\hbar} \begin{bmatrix} 0 & \exp(-\frac{i}{\hbar} \Delta E t)\Omega_1^* & 0 \\ \exp(\frac{i}{\hbar} \Delta E t)\Omega_1 & 0 & \exp(\frac{i}{\hbar} \Delta E t)\Omega_2^* \\ 0 & \exp(-\frac{i}{\hbar} \Delta E t)\Omega_2 & 0 \end{bmatrix} H(t) \phi. \] (3.3)

\( H(t) \) is periodic with period \( \Delta t = \frac{2\pi}{\Delta E} \). After \( \Delta t \), the system evolution is
\[ \phi(\Delta t) = (I + \int_0^{\Delta t} H(\sigma) d\sigma + \int_0^{\Delta t} \int_0^{\sigma_1} H(\sigma_1)H(\sigma_2)d\sigma_2d\sigma_1 + \ldots)\phi(0). \] (3.4)

The first integral averages to zero, while the second integral
\[ \int_0^{\Delta t} \int_0^{\sigma_1} H(\sigma_1)H(\sigma_2)d\sigma_2d\sigma_1 = \frac{1}{2} \int_0^{\Delta t} \int_0^{\sigma_1} [H(\sigma_1),H(\sigma_2)]d\sigma_2d\sigma_1. \] (3.5)

Evaluating it explicitly, we get for our system that second order integral is
\[ \frac{-i\Delta t}{\hbar} \begin{bmatrix} 0 & \Omega_1^* & \Omega_2^* \\ \Omega_1 & 0 & 0 \\ \Omega_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 - E_2 \\ E_2 - E_1 \\ 0 \end{bmatrix}. \] (3.6)

Thus we have created an effective Hamiltonian
\[ \begin{bmatrix} 0 & \Omega_1^* & \Omega_2^* \\ \Omega_1 & 0 & 0 \\ \Omega_2 & 0 & 0 \end{bmatrix}. \] (3.7)
which couples level $|1\rangle$ and $|3\rangle$ and drives transition between them at rate $\mathcal{M} = \frac{\Omega_1 \Omega_2}{(E_1 - E_2)}$.

### 3.2 Coulomb Potential and Møller Scattering

In Møller scattering, electrons with momentum $p_1$ and $q_1$ exchange photon with momentum $k$ and scatter to new momentum states $p_2$ and $q_2$. Observe the virtual particle four momentum is $k$. The Feynman diagram for the process is in 3.2. There are two three level systems associated with this process. Let $P = p_1 + q_1$.

![Feynman diagram](image)

**Fig. 3.2** Fig. shows the Feynman diagram for the Møller scattering and its corresponding three level system. The electron with momentum $p_1$ emits (absorbs) a photon and scatters to momentum $p_2$, the photon is absorbed (emitted) by electron with momentum $q_1$ which scatters to momentum $q_2$.

In figure 3.2A, we have the first three level system where the electron with momentum $p_1$ is annihilated, a electron of momentum $p_2$ is created and a photon of momentum $k = p_1 - p_2$ is created. Subsequently, the electron with momentum $q_1$...
is annihilated, a electron of momentum $q_2$ is created and photon of momentum $k$ is annihilated. The amplitude for this process is

$$\Omega_1 = \frac{C}{\sqrt{2E_k}} \bar{u}(p_2)\gamma^\nu \epsilon^\nu_v(k)u(p_1), \quad (3.8)$$

$$\Omega_2 = \frac{C}{\sqrt{2E_k}} \bar{u}(q_2)\gamma^\nu \epsilon^\nu_v(k)u(q_1), \quad (3.9)$$

$$E_1 - E_2 = E_{p_1} - E_{p_2} - E_k = k_0 - E_k, \quad (3.10)$$

$$\mathcal{M}_1 = \frac{\Omega_1 \Omega_2}{(E_1 - E_2)}. \quad (3.11)$$

Similarly we have another three level system, fig 3.2B in which $q_1$ emits photon with momentum $-k$ and $p_1$ absorbs it. This gives

$$\Omega_1 = \frac{C}{\sqrt{2E_k}} \bar{u}(q_2)\gamma^\nu \epsilon^\nu_v(k)u(q_1), \quad (3.12)$$

$$\Omega_2 = \frac{C}{\sqrt{2E_k}} \bar{u}(p_2)\gamma^\nu \epsilon^\nu_v(k)u(p_1), \quad (3.13)$$

$$E_1 - E_2 = E_{q_1} - E_{q_2} - E_k = -(k_0 + E_k), \quad (3.14)$$

$$\mathcal{M}_2 = \frac{\Omega_1 \Omega_2}{(E_1 - E_2)}. \quad (3.15)$$

where we used conservation of energy $E_{p_2} - E_{p_1} = E_{q_1} - E_{q_2}$. When we add the two amplitudes, we get for $k = p_1 - p_2$ and $k^2 = k_\mu k^\mu$, the total amplitude is

$$\mathcal{M} = \frac{\Omega_1 \Omega_2}{k^2} = C^2 \frac{\bar{u}(q_2)\gamma^\nu \epsilon^\nu_v(k)u(q_1) \bar{u}(p_2)\gamma^\nu \epsilon^\nu_v(k)u(p_1)}{k^2} \quad (3.16)$$

We can now sum over photon polarization $\epsilon$. We discuss this in a moment. Lets digress to Lorentz Invariance.

### 3.2.1 Lorentz Invariance

We want expression in 3.16 to be lorentz invariant as explained below. The term

$$\bar{u}(p_2)\gamma^\nu \epsilon^\nu_v(k)u(p_1) \quad (3.17)$$

is not lorentz invariant but

$$\bar{u}(p_2)\gamma^\nu \epsilon^\nu_v(k)u(p_1) \quad (3.18)$$

is. In 3.16 we multiple and divide as below
Here $E_T$ is total energy of incoming electrons. Then observe in the center of mass frame where $p_1 = q_1 = p_2 + q_2$, we have factor $\sqrt{E_{p_1}}$ in numerator and $E_T$ in denominator cancel. However in a frame different from center of mass, the numerator is Lorentz invariant (we discuss sum of electron polarization in a bit). The denominator transforms as $E_T \rightarrow E_T \gamma$ now one of the $\gamma$ cancels the fact that $C^2$ has $V$ the volume which transforms as $V \rightarrow \gamma V$ and hence cancels one $\gamma$. Then we are getting that $M \rightarrow M \gamma$ which is what we want because the transformed time $t$ is dilated version of the one $t'$ in the rest frame, i.e, $t = \frac{t'}{\gamma}$. Hence $(M \gamma)(\frac{t'}{\gamma}) = Mt'$ Now lets sum over photon polarization states in (3.19). To fix ideas say the photon momentum is in $z$ direction. Then we have two transverse polarization states $\varepsilon_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\varepsilon_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Substituting the two states get

$$M = \frac{4C^2}{E_T^2} \tilde{u}(q_2)\gamma^\nu u(q_1) \tilde{u}(p_2)\gamma^\nu u(p_1) + \tilde{u}(q_2)\gamma^\nu u(q_1) \tilde{u}(p_2)\gamma^\nu u(p_1)$$ (3.20)

If we make a lorentz transformation which is say a boost along $x$ direction. Then photon momentum changes to one in $x-z$ plane and the above expression transforms with $\varepsilon_1 \rightarrow \begin{bmatrix} \sin \gamma \\ \cos \gamma \\ 0 \\ 0 \end{bmatrix}$ and $\varepsilon_2$ as above. Observe $\varepsilon_1$ has time-like polarization. But this means in the boosted frame to get the correct (Lorentz invariant) answer, in deciding photon polarization states, we just didn’t choose transverse states, instead we choose for $\varepsilon_1$ a state with time like polarization. It means, we ended up making gauge choice to get the same answer. Hence we get lorentz invariance at cost of choice of special gauge. This is unsatisfactory. We want an answer that is independent of choice of gauge. To rectify this situation, we propose a propagator that is intrinsically lorentz invariant.

In Eq. (3.20) suppose we make a boost by a very large velocity in direction of $z$. Then in this frame all momentum are along $z$ direction all in relativistic limit. And as a result we have in this frame

$$\tilde{u}(q_2)\gamma^\nu u(q_1) \tilde{u}(p_2)\gamma^\nu u(p_1) - \tilde{u}(q_2)\gamma^\nu u(q_1) \tilde{u}(p_2)\gamma^\nu u(p_1) = 0$$ (3.21)

Then lets add above zero to Eq. (3.20) to get in this boosted frame,
\[ \mathcal{M} = \frac{4C^2}{E_T^2} \bar{u}(q_2) \gamma^\mu u(q_1) \; g_{\mu \nu} \; \bar{u}(p_2) \gamma^\nu u(p_1) \]  
(3.22)

where metric

\[ g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]  
(3.23)

Now how do we see that propagator is Lorentz invariant. The form we have is

\[
\begin{bmatrix}
\bar{u}(q_2) \gamma^0 u(q_1) \\
\bar{u}(q_2) \gamma^a u(q_1) \\
\bar{u}(q_2) \gamma^0 u(q_1) \\
\bar{u}(q_2) \gamma^a u(q_1)
\end{bmatrix} \Lambda' \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Lambda
\]

Lorentz transformation makes the form

\[ \begin{bmatrix}
\bar{u}(q_2) \gamma^0 u(q_1) \\
\bar{u}(q_2) \gamma^a u(q_1) \\
\bar{u}(q_2) \gamma^0 u(q_1) \\
\bar{u}(q_2) \gamma^a u(q_1)
\end{bmatrix} \Lambda' \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Lambda
\]

but \( \Lambda' g \Lambda = g \) which makes the form Lorentz invariant.

Let us calculate in the center of mass frame,

\[ \mathcal{M}_k = \frac{4C^2}{E_T^2} \bar{u}(q_2) \gamma^0 u(q_1) \; g_{\mu \nu} \; \bar{u}(p_2) \gamma^\nu u(p_1) \]  
(3.26)

cancelling factors in numerator and denominator we get

\[ \mathcal{M}_k = C^2 \bar{u}(q_2) \gamma^0 u(q_1) \; g_{\mu \nu} \; \bar{u}(p_2) \gamma^\nu u(p_1) \]  
(3.27)

Now observe assuming momentum \( p_1, q_1, p_2, q_2 \) are non relativistic

\[
\bar{u}(p_2) \gamma^0 u(p_1) = \frac{1}{2} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \otimes \xi \otimes \sigma_a \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \otimes \xi = 0
\]  
(3.28)

\[
\bar{u}(p_2) \gamma^a u(p_1) = \frac{1}{2} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \otimes \xi \otimes 1 \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \otimes \xi = 1
\]  
(3.29)

Then observe \( q^2 = (E_{p_1} - E_{p_2})^2 - (\hbar c)^2 |k|^2 \sim -(\hbar c)^2 |k|^2 \) under non-relativistic approximation. Then we get

\[ \mathcal{M}_k = \frac{q^2}{\varepsilon_0 V} \frac{1}{|k|^2} \]  
(3.30)

This gives a scattering potential
3.3 Bhaba scattering

\[ V = \sum_k \mathcal{M}_k \exp(-ik(r_1 - r_2)) = \frac{q^2}{(2\pi)^3 \varepsilon_0} \int d^3k \frac{\exp(-ik(r_1 - r_2))}{|k|^2}. \]  

(3.31)

For \( r = r_1 - r_2 \), we have,

\[
\int d^3k \frac{\exp(-ik(r_1 - r_2))}{|k|^2} = 2\pi \int d|k| \int_0^\pi \exp(-i|k||r| \cos \theta) \sin \theta d\theta
\]

\[
= 4\pi \int_0^\infty \frac{\sin |k||r|}{|k|} d|k| = \frac{2\pi^2}{|r|}
\]

\[ V = \sum_k \mathcal{M}_k \exp(-ik(r_1 - r_2)) = \frac{q^2}{4\pi\varepsilon_0|r|}. \]  

(3.32)

The familiar Coulomb potential.

### 3.3 Bhaba scattering

In quantum electrodynamics, Bhaba scattering is the electron-positron scattering process:

\[ e^+e^- \rightarrow e^+e^- \]  

(3.33)

Bhaba scattering is named after the Indian physicist Homi J. Bhabha. The Bhabha scattering rate is used as a luminosity monitor in electron-positron colliders.

How do we understand scattering of an electron and positron.

#### 3.3.1 Annihilation

See 3.4. An photon of momentum \( p + k \) comes and strikes filled sea of negative energy electrons and ejects a negative energy electron of momentum \(-k\) and creates a electron of momentum \( p \) and leaves behind missing momentum \(-k\) and missing charge \(-e\) or positron (hole) with momentum \( k \) and charge \( e \). The process is like photoelectric effect where a valence electron is ejected to a free electron however unlike photoelectric effect the above process is inelastic and doesnot proceed as described before. If we read this processs reverse then we have electron-positron pair of momentum \( p \) and \( k \) annihilate to form a photon of momentum \( p + k \). Denoting electron and positron helicity by \( s, s' \) and \( t, t' \) etc., the transition amplitude for this process is

\[ \Omega^\dagger = \frac{C}{\sqrt{2E_{p+k}}} \tilde{u}_s(p)\epsilon_\mu \gamma^\mu v_i(-k) \]  

(3.34)
absorbing the negative sign in Helicity $t$ we get

$$\Omega_1 = \frac{C}{\sqrt{2E_{p+k}}} \bar{u}_s(p) \gamma^\mu v_i(k)$$  \hspace{1cm} (3.35)

Now read the process in reverse and the transition amplitude for this process is

$$\Omega_1 = -\frac{C}{\sqrt{2E_{p+k}}} \bar{v}_i(k) \gamma^\mu \gamma^\nu u_s(p)$$  \hspace{1cm} (3.36)
Fig. 3.4 Above Fig. shows how a photon ejects a negative energy electron and creates a electron and positron (hole) pair.

The photon resulting from annihilation can now do ejection to create electron-positron or electron-hole pair with momentum $p'$ and $k'$ respectively with transition amplitude

$$\Omega_2 = \frac{C}{\sqrt{2E_{p+k}}} \bar{u}'(p')\gamma_\mu \rho_{\nu'}(k')$$

(3.37)

All this is depicted as a three level process in fig. 3.3B. The associated feynman diagram is in fig. 3.3A. The energy level difference between the ground and excited states

$$\Delta E_a = E_1 - E_2 = E_p + E_k - E_{p+k}.$$  (3.38)

where $E_p$, $E_k$ and $E_{p+k}$ are electron, positron and photon energies.

There is another three level process in fig. 3.3C associated with Feynman diagram in fig. is 3.3A. In this process, the negative energy electron just emits a photon with momentum $-(p' + k') = -(p + k)$ and a electron and positron with momentum $p'$ and $k'$. The amplitude of this process is $\Omega_2$ above. The emitted photon then combines with incoming electron and fills the incoming hole (vacancy) with amplitude same as $\Omega_1$. The energy level difference between the ground and excited states

$$\Delta E'_a = E_1 - E_2 = -(E_p + E_k + E_{p+k}).$$  (3.39)

where $E_p$, $E_k$ and $E_{p+k}$ are electron, positron and photon energies. Then the amplitude $\mathcal{M}_1$ of the Feynman diagram in fig. 3.3A is sum of three level process in fig. 3.3B and three level process in fig. 3.3C. Then
\[
M_a = \Omega_1 \Omega_2 \left( \frac{1}{E_p + E_k - E_{p+k}} - \frac{1}{E_p + E_k + E_{p+k}} \right) 
\]
(3.40)

\[
= \Omega_1 \Omega_2 \frac{2E_{p+q}}{(E_p + E_k)^2 - E_{p+k}^2} 
\]
(3.41)

If we denote four momentum \( q = p + q \), and \( q^2 = q_\mu q^\mu \). Then

\[
M_a = \Omega_1 \Omega_2 \frac{2E_{p+q}}{q^2} 
\]
(3.42)

\[
= -C^2 \bar{v}_l(k) \gamma^\mu u_s(p) \frac{\bar{u}_{s'}(p') \epsilon_{\mu \nu} \gamma^n v_{r'}(k')}{(p+k)^2} 
\]
(3.43)

Now as in previous section on Moeller scattering, we have to sum over photon polarization giving us

\[
M_a = C^2 \bar{v}_l(k) \gamma^\mu u_s(p) \frac{\bar{u}_{s'}(p') \gamma^n v_{r'}(k')}{(p+k)^2}. 
\]
(3.44)

### 3.3.2 Scattering

There is one more Feynman diagram in fig. 3.3D that contributes to this scattering process. There are also two three level processes fig. 3.3E and fig. 3.3F that contribute to this diagram. In fig. 3.3E a electron \( p \) scatters to \( p' \) giving a photon of momentum \( q = p - p' \). This happens with amplitude

\[
\Omega_3 = C \sqrt{2E_q} \bar{u}_{s'}(p') \epsilon^\nu \gamma^n u_s(p) 
\]
(3.45)

and this photon then moves a negative energy (valence) electron to the hole spot of incoming positron and in the process create a new hole (positron). This happens with amplitude

\[
\Omega_4 = \frac{C}{\sqrt{2E_q}} C \bar{v}_l(k) \epsilon_{\nu} \gamma^n v_{r'}(k') 
\]
(3.46)

The difference in energies of ground and excited state is

\[
\Delta E_b = E_1 - E_2 = E_p - E_{p'} - E_{p-p'}. 
\]
(3.47)

In fig. 3.3F a valence electron scatters to fill the vacancy \( k \) and create new vacancy at \( k' \) giving a photon of momentum \( k - k' \). This happens with amplitude

\[
\Omega_5 = \frac{C}{\sqrt{2E_q}} \bar{v}_l(k) \epsilon^\nu \gamma^n v_{r'}(k') 
\]
(3.48)
This happens with amplitude

$$\Omega_4 = \frac{C}{\sqrt{2E_q}} \bar{u}_\nu(p') \epsilon_\nu \gamma^\mu u_\mu(p)$$  \hspace{1cm} (3.49)

$$\Delta E_b = E_1 - E_2 = E_{k} - E_{k'} - E_{k-k'}.$$  \hspace{1cm} (3.50)

Then the amplitude $\mathcal{M}_2$ of the Feynman diagram in fig. 3.3D is sum of three level process in fig. 3.3E and three level process in fig. 3.3F. Then

$$\mathcal{M}_2 = \Omega_3 \Omega_4 \left( \frac{1}{E_p - E_{p'} - E_{p-p'}} + \frac{1}{E_{k} - E_{k'} - E_{k-k'}} \right)$$  \hspace{1cm} (3.51)

$$\mathcal{M}_2 = \Omega_3 \Omega_4 \left( \frac{1}{E_p - E_{p'} - E_{p-p'}} - \frac{1}{E_{p} - E_{p'} + E_{p-p'}} \right)$$  \hspace{1cm} (3.52)

$$\mathcal{M}_2 = \Omega_3 \Omega_4 \left( \frac{2E_{p-p'}}{(E_{p} - E_{p'} - E_{p-p'})^2 - E_{p-p'}^2} \right)$$  \hspace{1cm} (3.53)

If we denote four momentum $q = p - p'$, and $q^2 = q_\mu q^\mu$. Then

$$\mathcal{M}_2 = \Omega_3 \Omega_4 \frac{2E_{p-p'}}{q^2}$$  \hspace{1cm} (3.54)

$$\mathcal{M}_2 = C^2 \frac{\bar{u}_\nu(p') \epsilon_\nu \gamma^\mu u_\mu(p) \bar{v}_\nu(k) \gamma^\mu v_\mu(k')}{{q}^2}$$  \hspace{1cm} (3.55)

Now as in previous section on Moeller scattering, we have to sum over photon polarization giving us

$$\mathcal{M}_4 = -C^2 \frac{\bar{u}_\nu(p') \gamma^\mu u_\mu(p) \bar{v}_\nu(k) \gamma^\mu v_\mu(k')}{(p-p')^2}.$$  \hspace{1cm} (3.56)

The total amplitude then is

$$\mathcal{M} = C^2 \left( - \frac{\bar{u}_\nu(p') \gamma^\mu u_\mu(p) \bar{v}_\nu(k) \gamma^\mu v_\mu(k')}{(p-p')^2} + \frac{\bar{v}_\nu(k) \gamma^\mu u_\mu(p) \bar{u}_\nu(p') \gamma^\mu v_\mu(k')}{(p+k)^2} \right) = C^2 \mathcal{N}. $$  \hspace{1cm} (3.57)

### 3.3.3 Cross-section

Fig. (3.5)A shows the schematic of electron positron each of volume $V = l^3$ colliding head on. We can ask what should be the smallest density or the cross-section area $A = l^2$ for the two to scatter at an angle $\theta$ with probability 1, when they collide. This is called differntial cross-section. We have calculated $\mathcal{M}$ the scattering amplitude,
in center of mass frame. Let $M(p_i)$ denote this as function of outgoing momenta $p_i$. Then by Fermi Golden rule the probability of scattering $P$ is given by

$$\frac{dP}{dt} = \sum_i |M(p_i)|^2 \frac{\bar{h} \Delta E}{\hbar}$$  \hspace{1cm} (3.58)

where $\Delta E$ is the energy width of the tessellation of the momentum space volume as shown in Fig. (3.5)C. Note $|M(p_i)|^2$ carries with it a factor $C^4$ which has in it $(\hbar c)^4$. Let $\frac{1}{\nu} = \frac{d^4k}{(2\pi)^3}$ or $\nu = \frac{(2\pi)^3}{(\hbar c)^4}$. With $E = \sqrt{p^2 + m^2}$ we get $\Delta E = \frac{e \Delta p}{c}$. Then converting sum in 3.58 to integral we get

$$\frac{dP}{dt} = \frac{e^4 c}{\epsilon_0^2 (2\pi)^3} \int |N(p)|^2 \frac{d^3\Omega}{\Delta E}$$  \hspace{1cm} (3.59)

Then it takes $\Delta t = \frac{l}{c}$ for the packets to cross each other and it this time we want

$$\frac{dP}{dt} \frac{l}{c} = 1$$  \hspace{1cm} (3.60)

or we get using $\Delta E$

$$\sigma = l^2 = \frac{e^4}{\epsilon_0^2 (2\pi)^3} \frac{E^2}{\hbar^2} \int |N(p)|^2 d\Omega$$  \hspace{1cm} (3.61)

or

$$\frac{d\sigma}{d\Omega} = \frac{e^4 E^2}{\epsilon_0^2 (2\pi)^3} |N(p)|^2$$  \hspace{1cm} (3.62)

This is called differential cross-section.
Now we calculate differential cross section by evaluating

\[ |\mathcal{N}|^2 = |\mathcal{M}_a + \mathcal{M}_s|^2 \] (3.63)

Infact we evaluate unpolarized cross-section which is to say we average over all possible helicities to get

\[
\sum_{s,s',t,t'} |\mathcal{N}|^2 = \sum_{s,s',t,t'} |\mathcal{M}_a|^2 + \sum_{s,s',t,t'} |\mathcal{M}_s|^2 + \sum_{s,s',t,t'} \mathcal{N}_a \mathcal{N}_s^* + \mathcal{N}_s \mathcal{N}_a^*
\]

### 3.3.4 Relativistic limit

Let's evaluate the unpolarized cross-section

\[
\frac{1}{4} \sum_{s,s',t,t'} |\mathcal{N}|^2.
\] (3.64)

Recall

\[
\mathcal{N}_s = \frac{-\bar{u}_s(p') \gamma^0 u_s(p) \bar{v}_t(k) \gamma^0 v_t(k')}{(p - p')^2}
\] (3.65)

Let electron and positron approach each other along \(z\) and \(-z\) direction respectively. Under relativistic limit helicity 1 electron and positron are

\[
u(p) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \quad v(k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix};
\]

\[
u(p') = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} ; \quad v(k') = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix};
\]

Under relativistic limit helicity \(-1\) electron and positron are

\[
u(p) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} ; \quad v(k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix};
\]

\[
u(p') = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix} ; \quad v(k') = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix};
\]

Now observe in 3.65, we get zero if we switch either electron and positron helicity. Furthermore under relativistic limit we get

\[
(p - p')^2 \sim -2p.p' \sim -2E^2(1 - \cos \theta).
\] (3.66)
Then substituting all helicities in Eq. 3.64, we get
\[ \frac{1}{4} \sum_{s,s',t,t'} |\mathcal{N}_s|^2 = \frac{1}{8E^4}. \]

Now let's evaluate
\[ \frac{1}{4} \sum_{s,s',t,t'} |\mathcal{N}_a|^2. \]  
(3.67)

Recall
\[ \mathcal{N}_a = \bar{v}_r(k) \gamma_\mu u_s(p) \bar{u}_{r'}(p') \gamma_\mu v_{r'}(k') \]  
(3.68)

Now observe under relativistic limit, in 3.68, we get zero if incoming or outgoing pair has same helicity. Then substituting all helicities in Eq. 3.67, we get
\[ \frac{1}{4} \sum_{s,s',t,t'} |\mathcal{N}_a|^2 = \frac{(1 - \cos \theta)^2}{4E^4}. \]

Finally evaluating
\[ \frac{1}{4} \sum_{s,s',t,t'} \mathcal{N}_a \mathcal{N}_a^* + \mathcal{N}_a^* \mathcal{N}_a = -\frac{(1 - \cos \theta)}{4E^4}. \]  
(3.69)

where we have only two terms, incoming or outgoing pair has same helicity and helicity cannot switch from incoming to outgoing. Adding everything we get
\[ \frac{1}{4} \sum_{s,s',t,t'} |\mathcal{N}|^2 = \frac{(1 - \cos \theta)^2 + \cos^2 \theta}{8E^4}. \]  
(3.70)

For \( s = E^2 \), we get
\[ \frac{d\sigma}{d\Omega} = \frac{e^4 (1 - \cos \theta)^2 + \cos^2 \theta}{32 \pi^2} \]  
(3.71)

### 3.4 Compton scattering

Compton scattering is the inelastic scattering of a photon with an electrically charged particle, first discovered in 1923 by Arthur Compton [?]. This scattering process is of particular historical importance as classical electromagnetism is insufficient to describe the process; a successful description requires us to take into account the particle-like properties of light. Furthermore, the Compton scattering of an electron and a photon is a process that can be described to a high level of precision by QED.
In Compton scattering an electron and photon with momentum $p$ and $k$ respectively scatter into momentum $p'$ and $k'$ respectively. We want to calculate the amplitude for this scattering.

There are two Feynman diagrams that show mechanism of Compton scattering. They are shown in Fig. 3.6. We can associate each of these with two three level diagrams as shown in Fig. 3.7.

Consider Feynman diagram A in Fig. 3.6, where a electron of momentum $p$ and photon of momentum $k$ are annihilated to give an electron of momentum $q = p + k$ which is then annihilated to create electron and photon with momentum $p'$ and $k'$. This correspond to three level system Fig. 3.7 A. The scattering amplitude for this system is as follows.
\[ \Omega_1 = \frac{C}{\sqrt{2E_k}} \bar{u}_s(p+k) \gamma^{\mu} \epsilon_{\nu}(k) u(p), \quad (3.72) \]

\[ \Omega_2 = \frac{C}{\sqrt{2E_{k'}}} \bar{u}(p') \gamma^{\mu} \epsilon^{*}_{\mu}(k') u_s(p+k), \quad (3.73) \]

\[ E_1 - E_2 = q_0 - E_q = E_p + E_k - E_{p+k}, \quad (3.74) \]

\[ \mathcal{M}_{\lambda a}^{\pm} = \frac{\Omega_1 \Omega_2}{E_1 - E_2}. \quad (3.75) \]

where \( E_p = \sqrt{(|p|c)^2 + m^2} \) and \( E_k = |k|c \). Summing over electron polarization we get
We first create electron and photon with momentum $p$ and $q$ respectively alongside a positron with momentum $q = -(p' + k') = -(p + k)$ and then annihilate electron and photon with momentum $p$ and $k$ alongside a positron with momentum $-(p + k)$. The scattering amplitude for this system is as follows

\[ \mathcal{M}_a = \frac{C^2}{2\sqrt{E_k E_k'}} \bar{u}(p') \gamma^\mu e^\mu_\nu(k') \sum u_s(p + k) \bar{u}_s(p + k) \frac{\gamma^\nu e_\nu(k)u(p)}{E_q(q_0 - E_q)}. \]  

(3.76)

There is an associated three level diagram with this as shown in 3.7 B, where we first create electron and photon with momentum $p'$ and $k'$ respectively alongside a positron with momentum $q = -(p' + k') = -(p + k)$ and then annihilate electron and photon with momentum $p$ and $k$ alongside a positron with momentum $-(p + k)$. The scattering amplitude for this system is as follows

\[ \Omega_1 = \frac{C}{\sqrt{2E_k}} \bar{u}(p') \gamma^\mu e^\mu_\nu(k')u_s(p + k), \]  

(3.77)

\[ \Omega_2 = \frac{C}{\sqrt{2E_k}} \bar{u}_s(p + k) \gamma^\nu e_\nu(k)u(p), \]  

(3.78)

\[ E_1 - E_2 = -(q_0 + E_q) = -E_{p+k} - (E_p + E_k), \]  

(3.79)

\[ \mathcal{M}_{ib} = \frac{\Omega_1 \Omega_2}{E_1 - E_2}. \]  

(3.80)

Summing over electron polarization we get

\[ \mathcal{M}_{ib} = -\frac{C^2}{2\sqrt{E_k E_k'}} \bar{u}(p') \gamma^\mu e^\mu_\nu(k') \sum u_s(p + k) \bar{u}_s(p + k) \frac{\gamma^\nu e_\nu(k)u(p)}{E_q(q_0 + E_q)}. \]  

(3.81)

Adding the two amplitudes $\mathcal{M} = \mathcal{M}_a + \mathcal{M}_{ib}$, we get

\[ \mathcal{M}_1 = \frac{C^2}{\sqrt{E_p E_p' E_k E_k'}} \frac{\bar{u}(p') \gamma^\mu e^\mu_\nu(k') \sum u_s(p + k) \bar{u}_s(p + k) \gamma^\nu e_\nu(k)u(p)}{q^2 - m_0^2}, \]

\[ = \frac{C^2}{2\sqrt{E_p E_p' E_k E_k'}} \frac{\bar{u}(p') \gamma^\mu e^\mu_\nu(k')(q + m_0) \gamma^\nu e_\nu(k)u(p)}{q^2 - m_0^2}. \]  

(3.82)

We made use of identity $\sum u_s(q) \bar{u}_s(q) = \frac{q + m_0}{2E_q}$, where $q = q_j \gamma^j$ (c is implicit). We assume we are in a high energy center of mass frame. Which implies $E_p \sim E_k$ and we can write

\[ \mathcal{M}_1 = \frac{C^2}{E_T^2} \frac{\bar{u}(p') \gamma^\mu e^\mu_\nu(k')(p + k + m_0) \gamma^\nu e_\nu(k)u(p)}{(p + k)^2 - m_0^2}. \]  

(3.83)

where $E_T = E_p + E_k$. Now observe $\mathcal{M}_1$ is lorentz invariant amplitude.

Now consider Feynman diagram B in Fig. 3.6, where a electron of momentum $p$ is annihilated and photon of momentum $k'$ is created to give an electron of momentum $q = p' - k'$ which is then annihilated along-with the photon of momentum $k$ to create electron with momentum $p'$. This correspond to three level system Fig. 3.7 C. The scattering amplitude for this system is as follows

\[ \mathcal{M}_1 = \frac{C^2}{E_T^2} \frac{\bar{u}(p') \gamma^\mu e^\mu_\nu(k')(p + k + m_0) \gamma^\nu e_\nu(k)u(p)}{(p + k)^2 - m_0^2}. \]  

(3.83)
\[ \Omega_1 = \frac{C}{\sqrt{2E_k'}} \bar{u}(p-k') \gamma^\mu e_\mu^*(k') u(p), \]  
(3.84)

\[ \Omega_2 = \frac{C}{\sqrt{2E_k}} \bar{u}(p') \gamma^\nu e_\nu(k) u_s(p-k'), \]  
(3.85)

\[ E_1 - E_2 = q_0 - E_q = E_p - E_{p-k'} - E_{k'}, \]  
(3.86)

\[ \mathcal{M}_{2a} = \frac{\Omega_1 \Omega_2}{E_1 - E_2}. \]  
(3.87)

Summing over electron polarization we get

\[ \mathcal{M}_{2a} = \frac{C^2}{2 \sqrt{E_k E_{k'}}} \bar{u}(p') \gamma^\nu e_\nu(k) \sum_s u_s(p-k') \bar{u}_s(p-k') \gamma^\mu e_\mu^*(k') u(p). \]  
(3.88)

There is an associated three level diagram with this as shown in 3.7 D, where we first create electron and annihilate photon with momentum \( p' \) and \( k \) respectively alongside creating a positron with momentum \( -(p-k') = -(p'-k) \) and then annihilate electron and create photon with momentum \( p \) and \( k' \) alongside annihilate positron with momentum \( -(p'-k') \).

The scattering amplitude for this system is as follows

\[ \Omega_1 = \frac{C}{\sqrt{2E_k'}} \bar{u}(p') \gamma^\nu e_\nu(k) u_s(p-k'), \]  
(3.89)

\[ \Omega_2 = \frac{C}{\sqrt{2E_k}} \bar{u}_s(p-k') \gamma^\mu e_\mu^*(k') u(p), \]  
(3.90)

\[ E_1 - E_2 = -(q_0 + E_q) = -E_{p-k'} - E_p + E_{k'}, \]  
(3.91)

\[ \mathcal{M}_{2b} = \frac{\Omega_1 \Omega_2}{E_1 - E_2}. \]  
(3.92)

Summing over electron polarization we get

\[ \mathcal{M}_{2b} = \frac{C^2}{2 \sqrt{E_k E_{k'}}} \bar{u}(p') \gamma^\nu e_\nu(k) \sum_s u_s(p-k') \bar{u}_s(p-k') \gamma^\mu e_\mu^*(k') u(p). \]  
(3.93)

Adding the two amplitudes \( \mathcal{M}_2 = \mathcal{M}_{2a} + \mathcal{M}_{2b} \), we get

\[ \mathcal{M}_2 = \frac{C^2}{\sqrt{E_p E_{p'}} E_k E_{k'}} \bar{u}(p') \gamma^\nu e_\nu(k) \sum_s u_s(p-k') \bar{u}_s(p-k') \gamma^\mu e_\mu^*(k') u(p), \]  
(3.94)

\[ \approx \frac{C^2}{2 \sqrt{E_p E_{p'} E_k E_{k'}}} \frac{\bar{u}(p') \gamma^\nu e_\nu(k) (p + m_0) \gamma^\mu e_\mu^*(k') u(p)}{q^2 - m_0^2}. \]  
(3.95)
\[ \mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 \]  

(3.96)

We have to calculate \( |\mathcal{M}|^2 \) to find cross-section. Before we do this we make few simplifications. Since \( p^2 = m^2 \) and \( k^2 = 0 \), the denominator of the propagators are

\[ (p + k)^2 - m^2 = 2p \cdot k; \quad (p - k')^2 - m^2 = -2p \cdot k' \]  

(3.97)

To simplify the numerators we use Dirac algebra.

\[ (p + m)\gamma^\nu \mathbf{u}(p) = (2p^\nu - \gamma^\mu (p - m))\mathbf{u}(p) \]  

(3.98)

\[ = 2p^\nu \mathbf{u}(p). \]  

(3.99)

\[ \mathcal{M} = \frac{C^2}{E_T^4} \varepsilon^\nu_\mu(k')\varepsilon_{\nu}(k)\mathbf{u}(p) \left\{ \frac{\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu + 2\gamma^\mu p^\nu}{2p \cdot k} + \frac{\gamma^\nu \gamma^\mu - \gamma^\mu \gamma^\nu + 2\gamma^\mu p^\nu}{2p \cdot k'} \right\} \mathbf{u}(p) \]  

(3.100)

3.4.1 The Klein-Nishina Formula

Computation of Compton scattering cross-section involves averaging \(|\mathcal{M}|^2\) over initial and final electron and photon polarization.

\[ \frac{1}{4} \sum |\mathcal{M}|^2 = \frac{C^4}{4E_T^4} 8\mu \rho \varepsilon_{\nu} \varepsilon_{\sigma} \cdot tr\{ \quad (p' + m) \left[ \frac{\gamma^\mu \gamma^\nu + 2\gamma^\mu p^\nu}{2p \cdot k} + \frac{\gamma^\nu \gamma^\mu - 2\gamma^\nu p^\mu}{2p \cdot k'} \right] \right\} \cdot (p + m) \left[ \frac{\gamma^\rho \gamma^\sigma + 2\gamma^\rho p^\sigma}{2p \cdot k} + \frac{\gamma^\sigma \gamma^\rho - 2\gamma^\sigma p^\rho}{2p \cdot k'} \right] \}

\[ \frac{1}{4} \sum |\mathcal{M}|^2 = \frac{C^4}{4E_T^4} \left\{ \frac{I}{(2p \cdot k)^2} + \frac{II}{(2p \cdot k)(2p \cdot k')} + \frac{III}{(2p \cdot k)(2p \cdot k')} + \frac{IV}{(2p \cdot k)^2} \right\} \]

Where \( I, II, III, IV \) are traces. First of the traces is

\[ I = tr \left[ (p' + m)(\gamma^\mu \gamma^\nu + 2\gamma^\mu p^\nu)(p + m)(\gamma_\mu \gamma_\nu + 2\gamma_\mu p_\nu) \right] \]

Traces with odd \( \gamma \) vanish. Evaluating these traces gives

\[ \frac{1}{4} \sum |\mathcal{M}|^2 = \frac{2C^4}{E_T^4} \left\{ \frac{p \cdot k'}{p \cdot k} + \frac{p \cdot k}{p \cdot k'} + 2m^2 \left( \frac{1}{p \cdot k} - \frac{1}{p \cdot k'} \right) + m^4 \left( \frac{1}{p \cdot k} - \frac{1}{p \cdot k'} \right)^2 \right\} \]
3.4.2 Cross-section

Fig. 3.8 Above Fig. A shows Compton scattering in lab frame. Fig. B shows Compton scattering in center of mass frame.

Let’s calculate the cross section in lab frame, Fig. 3.8A. The incoming photon with \( k = (\omega, \omega^2) \) strikes an electron and rest and scatters to state

\[ k' = (\omega', \omega' \cos \theta, 0, \omega' \sin \theta) \]

and electron scatters to state \( p' = (E', p') \).

\[
m^2 = p'^2 = (p + k - k')^2 = p^2 + 2p \cdot (k - k') - 2k' \cdot k'
\]

(3.101)

\[
m^2 = m^2 + 2m(\omega - \omega') - 2\omega \omega' (1 - \cos \theta),
\]

(3.102)

\[
\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1 - \cos \theta}{m}.
\]

(3.103)

In terms of wavelength

\[
\lambda' - \lambda = \frac{h}{mc} (1 - \cos \theta).
\]

(3.104)

\[
E_T = E' + \omega' = \sqrt{\omega^2 + \omega'^2 - 2\omega \omega' \cos \theta + m^2 + \omega'}
\]

(3.105)
In lab frame we assume $\omega \ll m$. Then $\frac{\omega'}{\omega} \sim 1$. Then in this nonrelativistic frame we replace $E^2_T$ by $4E_kE'_kE_pE'_p \sim 4\omega\omega'm^2$. Please refer to Eq. (3.82) and (3.83) and the approximation therein.

$$\frac{\Delta E_T}{\Delta \omega'} \sim 1$$ (3.106)

$$\frac{1}{4} \sum |M|^2 = 2 \left[ \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2 \theta \right] \sim 2(1 + \cos^2 \theta)$$ (3.107)

Then cross section is

$$\frac{d\sigma}{d\cos \theta} = 2 \frac{\omega^2 \Delta \omega'}{4\pi^2 \Delta E_T} \left(1 + \cos^2 \theta \right) \sim \frac{(1 + \cos^2 \theta)}{8\pi^2 m^2}.$$ (3.108)

We now consider cross section in center of mass frame, Fig. 3.8B. The incoming photon with $k = (\omega, \omega\hat{z})$ collides with electron $p' = (E, -\omega\hat{z})$ and scatters to state $k' = (\omega, \omega \cos \theta, 0, \omega \sin \theta)$ and electron scatters to state $p' = (E, -k')$. Observe

$$p \cdot k = \omega(E + \omega)$$
$$p \cdot k' = \omega(E + \omega \cos \theta)$$
$$E^2 = m^2 + \omega^2$$

We assume high energy frame $E \gg m$. Under this assumption we get

$$\frac{1}{4} \sum |M|^2 = \frac{2C^4}{E^4} \frac{p \cdot k}{p \cdot k'} = \frac{2C^4}{E^4} \frac{E + \omega}{E + \omega \cos \theta}$$ (3.109)

$$E_T = \omega + \sqrt{\omega^2 + m^2} \sim 2\omega$$ (3.110)

$$\frac{\Delta E_T}{\Delta \omega} = 2$$ (3.111)

$$\frac{d\sigma}{d\cos \theta} = \frac{2e^4}{(E + \omega)^4} \frac{\omega^2 \Delta \omega}{4\pi^2 \Delta E_T} \frac{E + \omega}{E + \omega \cos \theta} \sim \frac{1}{64\pi^2} \frac{1}{\omega(E + \omega \cos \theta)}.$$ (3.112)
3.5 Vacuum Polarization

Recall Møller scattering. Pair of electrons scatter by exchanging a photon as shown in Fig. 3.2. The phonon can give rise to electron positron pair which recombine again to give the phonon as shown in Fig. 3.9. The electron-positron pair that is created screen the charge of electrons and results in reduced Coulomb repulsion. We say the vacuum gets polarized. The resulting $\varepsilon_0$ increases so that $\frac{q^2}{\varepsilon_0}$ decreases.

How much is this polarization. We can compute it using the apparatus of QED. The creation of electron positron pair can be represented by an additional level 4, in energy level diagram of moeller scattering as shown in Fig. 3.9B. Lets focus on two levels 2 and 4.

Consider the Feynman diagram in Fig. 3.10 A, a photon with momentum $k$ spontaneously creates a electron-positron pair with momentum $p + k$ and $-p$ respectively, which recombine to give $k$ again. Creating a $-p$ positron is same as annihilate a negative energy electron with momentum $p$. The two levels shown in Fig. 3.10 B,
3.5 Vacuum Polarization

Fig. 3.10 Fig. shows how to model vacuum polarization as a two level system.

with lower level 2 comprising of incoming momentum \( p,k \) and higher level 4, the outgoing electron with momentum \( p+k \). The transition rate \( \Omega \) from 2 to 4 is given by

\[
\Omega = \frac{C}{\sqrt{2E_k}} \bar{u}(p+k)\gamma^\nu\bar{e}_\nu(k)\nu(p),
\]  

(3.113)

There is an concomitant two level system with this where we first create an electron, positron and photon with momentum \( p, -(p+k), k \) respectively. This is shown in Fig. 3.10 C, where again creating a \(-(p+k)\) positron is same as annilitate a negative energy electron with momentum \( p+k \). The transition rate \( \Omega \) from 1 to 2 is

\[
\Omega = \frac{C}{\sqrt{2E_k}} \bar{u}(p)\gamma^\nu\bar{e}_\nu(k)\nu(p+k),
\]  

(3.114)

What is the effect of this fourth level in Fig. 3.9B.

\[
\psi = -i\frac{\hbar}{\hbar} \begin{bmatrix} E_1 & 0 & \Omega_1^* & 0 \\ 0 & E_3 & \Omega_2 & 0 \\ \Omega_1 & \Omega_2^* & E_2 & \Omega_4^* \\ 0 & 0 & \Omega & E_4 \end{bmatrix} \psi.
\]

(3.115)

We transform,

\[
\phi(t) = \exp(i\frac{\hbar}{\hbar} t) \begin{bmatrix} E_1 & 0 & \Omega_1^* & 0 \\ 0 & E_3 & \Omega_2 & 0 \\ \Omega_1 & \Omega_2^* & E_2 & \Omega_4^* \\ 0 & 0 & 0 & E_4 \end{bmatrix} \phi(t).
\]

(3.116)
This gives for $\Delta E = E_2 - E_4$

\[
\dot{\phi} = -\frac{i}{\hbar} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \exp\left(\frac{i}{\hbar}\Delta E\, t\right) \Omega \ast \\ 0 & 0 & \exp\left(\frac{i}{\hbar}\Delta E\, t\right) \Omega \end{bmatrix} \phi. \tag{3.117}
\]

$H(t)$ is periodic with period $\Delta t = \frac{2\pi}{\Delta E}$. After $\Delta t$, the system evolution is

\[
\phi(\Delta t) = \exp\left(-\frac{i}{\hbar} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E'_2 \\ 0 & 0 & E'_4 \end{bmatrix} \Delta t\right) \phi(0), \tag{3.118}
\]

with $E'_2, E'_4$ as calculated in description of energy shifts. Then

\[
\psi(\Delta t) = \exp\left(-\frac{i}{\hbar} \begin{bmatrix} E_1 & 0 & \Omega'_1 & 0 \\ 0 & E_3 & \Omega_2 & 0 \\ \Omega_1 & \Omega'_3 & E_2 & 0 \\ 0 & 0 & 0 & E_4 \end{bmatrix} \Delta t\right) \exp\left(-\frac{i}{\hbar} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E'_2 \\ 0 & 0 & 0 & E'_4 \end{bmatrix} \Delta t\right) \psi(0)
\]

\[
\sim \exp\left(-\frac{i}{\hbar} \begin{bmatrix} E_1 & 0 & \Omega'_1 & 0 \\ 0 & E_3 & \Omega_2 & 0 \\ \Omega_1 & \Omega'_3 & E_2 + E'_2 & 0 \\ 0 & 0 & 0 & E_4 + E'_4 \end{bmatrix} \Delta t\right) \psi(0)
\]

Thus we have removed coupling $\Omega$ and introduced energy shifts in $E_2$ and $E_4$. We can now eliminate level 2 and calculate transition amplitude between 1 and 3 and as before. What are these energy shifts $E'_2$ and $E'_4$.

The dynamics of this two level system is

\[
\psi = -\frac{i}{\hbar} \begin{bmatrix} E_2 & \Omega' \\ \Omega & E_4 \end{bmatrix} \psi. \tag{3.119}
\]

We proceed into the interaction frame of the natural Hamiltonian (system energies) by transformation

\[
\phi(t) = \exp\left(\frac{i}{\hbar} \begin{bmatrix} E_2 & 0 \\ 0 & E_4 \end{bmatrix} t\right) \psi(t). \tag{3.120}
\]

This gives for $\Delta E = E_4 - E_2$

\[
\dot{\phi} = -\frac{i}{\hbar} \begin{bmatrix} 0 & \exp\left(-\frac{i}{\hbar}\Delta E\, t\right) \Omega \ast \\ \exp\left(\frac{i}{\hbar}\Delta E\, t\right) \Omega \\ \end{bmatrix} \phi. \tag{3.121}
\]

$H(t)$ is periodic with period $\Delta t = \frac{2\pi}{\Delta E}$. After $\Delta t$, the system evolution is
\[ \phi(\Delta t) = (I + \int_{0}^{\Delta t} H(\sigma)d\sigma + \int_{0}^{\sigma_1} \int_{0}^{\Delta t} H(\sigma_1)H(\sigma_2)d\sigma_2d\sigma_1 + \ldots)\phi(0). \]  

(3.122)

The first integral averages to zero, while the second integral

\[ \int_{0}^{\Delta t} \int_{0}^{\sigma_1} H(\sigma_1)H(\sigma_2)d\sigma_2d\sigma_1 = \frac{1}{2} \int_{0}^{\Delta t} \int_{0}^{\sigma_1} [H(\sigma_1), H(\sigma_2)]d\sigma_2d\sigma_1. \]  

(3.123)

Evaluating it explicitly, we get for our system that second order integral is

\[ \frac{-i\Delta t}{\hbar} \begin{bmatrix} \Omega^* \Omega & 0 \\ \Omega \Omega^* & -\frac{\Omega^* \Omega}{E_2 - E_4} \end{bmatrix}. \]  

(3.124)

Thus we have created a shift of energy level 2 from \( E_2 \rightarrow E_2 + \frac{\Omega^* \Omega}{(E_2 - E_4)} \).

Now we donot have one fourth level but many corresponding to electron positron pairs of different momentum \( p \) as shown in Fig. 3.11. Then total \( E_2' = \sum_p E_2'(p) \), then we add energy shifts due to this.

![Fig. 3.11](image-url)  

**Fig. 3.11** Fig. shows how vacuum polarization is represented as additional levels when Möller scattering is represented as a three level system.

Now we calculate energy shifts \( E_2' \).

\[ E_2' = \frac{C^2}{2E_4(E_k - E_p - E_{p+k})} \tilde{v}(p)\gamma^\mu u(p+k)\bar{u}(p+k)\gamma_\mu v(p) \]
The shift due to *concomitant* process is

\[ E'_2 = -\frac{C^2}{2E_k(E_p + E_k + E_{p+k})} \bar{v}(p)\gamma^\mu u(p + k)\bar{u}(p + k)\gamma_\mu v(p) \]

adding the two shifts gives us

\[ E'_2 = \frac{C^2(E_p + E_{p+k})}{E_k((E_{p+k} + E_p)^2 - E_k^2)} g^{\mu\nu} \bar{v}(p)\gamma^\mu u(p + k)\bar{u}(p + k)\gamma_\nu v(p) \]

How does this shift effect the amplitude of Moeller scattering. The amplitude is

\[ \mathcal{M} = \frac{C^2}{E_{cm}E_k} g^{\mu\nu} \bar{u}(q_2)\gamma^\mu u(p_2)\bar{u}(q_1)\gamma^\nu u(p_1) \left( \frac{1}{E_{p_1} - E_{q_1} - E_k - E_k'} + \frac{1}{E_{q_1} - E_{q_2} - E_k - E_k'} \right) \]

\[ = \frac{C^2}{E_{cm}E_k} \bar{u}(q_2)\gamma^\mu u(p_2)\bar{u}(q_1)\gamma^\nu u(p_1) \left( \frac{E_k + E_k'}{q^2} \right) \]

Coming back to shift in Eq. (3.125), we evaluate it for large \( p \) states in the CM frame where \( p + k = 0 \), then in this frame both \( k, p \) states become relativistic and using the approximation that in CM frame \( \frac{E_p}{E_k} \sim 0 \) and \( \frac{E_k'}{E_k} \sim 1 \) we find

\[ E'_2 \sim \frac{C^2}{(E_{p+k} + E_p)^2 - E_k^2} \bar{v}(p)\gamma^\mu u(p + k)\bar{u}(p + k)\gamma_\mu v(p) \quad (3.125) \]

Now using \( E_{p+k} = m_0 \) in CM frame we find we can bound \( E'_2 \) by \( E_b \)

\[ E'_2 < E_b = \frac{C^2 m_0^2}{E_{p+k}((E_{p+k} + E_p)^2 - E_k^2)} \quad (3.126) \]

Observe the bound \( E_b \) transforms as *Lorentz amplitude*. We can evaluate \( E_b \) in our original frame. For large \( p \) we have

\[ E_b \sim \frac{C^2 m_0^2}{|p|^4} \quad (3.127) \]

Lets sum \( E_b \) over \( p \), for \( |p| > p_0 \) then we find the sum is convergent

\[ 4\pi \int \frac{1}{|p|^4} |p|^2 dp = 4\pi \int \frac{1}{|p|^2} dp = \frac{4\pi}{p_0} \quad (3.128) \]
3.6 Electron self energy

In last section, we talked about vacuum polarization, where a photon splits into an electron-positron pair and recombines. In this section, we discuss another QED process, the electron self energy. Hereby, an electron of momentum $p$ emits a photon and then reabsorbs it. This is shown in Fig. 3.12A. This process can be represented by two level diagrams as in Fig. 3.12B. In the first one, we have an electron with momentum $p$ emit an photon with momentum $k$ and subsequently reabsorb it. In second one, we have creation of a positron, electron and photon with momentum $-p$, $p-k$ and $k$ respectively and their subsequent annihilation.

![Diagram showing electron self energy](image)

**Fig. 3.12** Fig. A shows corrections to electron energy, where an electron emits and absorbs an photon. Fig. B shows two level diagrams for this process.

The transition rate $\Omega$ from level 1 to 2 in the first one is given by

$$\Omega = \frac{C}{\sqrt{2E_k}} \bar{u}(p-k) \gamma^\nu \epsilon^*_\nu(k) u(p), \quad (3.129)$$

The outgoing electron is the virtual particle, the CM frame is its rest frame. The transition rate $\Omega$ from 1 to 2 in the second process is given by

$$\Omega = \frac{C}{\sqrt{2E_k}} \bar{u}(p-k) \gamma^\nu \epsilon^*_\nu(k) u(p), \quad (3.130)$$

Now we can calculate shift of energy level 1 from $E_1 \rightarrow E_1 + \frac{\Omega \Omega^*}{(E_1 - E_2)}$ for large $k$. Using $\Omega$ from Eq. (3.129) and Eq. (3.130), we find, in CM frame where $p - k = 0$, $p$ and $k$ are relativistic, $\frac{E_p}{E_k} \sim 1$ and $\frac{E_{p-k}}{E_k} \sim 0$.
\[ E'_1(k) \propto \frac{1}{(E_k)} \left( \frac{1}{E_p - E_{p-k} - E_k} - \frac{1}{E_p + E_k + E_{p-k}} \right) \sim \frac{1}{(E_k + E_{p-k})^2 - E_p^2}. \] (3.131)

Now using \( E_{p-k} = m_0 \) in CM frame we find we can bound \( E'_1 \) by \( E_b \)

\[ E'_1 < E_b = \frac{C^2 m_0^2}{E_{p-k}((E_{p-k} + E_k)^2 - E_p^2)}. \] (3.132)

Observe the bound \( E_b \) transforms as Lorentz amplitude. We can evaluate \( E_b \) in our original frame. For large \( k \) we have

\[ E_b \sim \frac{C^2 m_0^2}{|k|^4} \] (3.133)

Lets sum \( E_b \) over \( k \), for \( |k| > k_0 \) then we find the sum is convergent

\[ 4\pi \int \frac{1}{|k|^4} |k|^2 dp = 4\pi \int \frac{1}{|k|^2} dp = \frac{4\pi}{k_0}. \] (3.134)

Thus \( E_1 \to E_1 + E'_1 \) can be written as correction to mass \( m_0 \to m_0 + m'(p) \). All corrections are finite so there is no problem.

### 3.7 Vertex Corrections

Consider the Feynman diagram in Fig. 3.13A. It shows moller scattering of incoming electron with momentum \( p_1 \) and a heavy particle with momentum \( r_1 \). Incoming electron emits a photon with momentum \( k \) that recombines with outgoing electron with momentum \( p_2 \). Fig. B shows an equivalent five level system. The incoming particles with momentum \( p_1, r_1 \) are at level 1. Emission of a photon with momentum \( k \) transits to level 2. Level 2, 3, 4 represent the Moller scattering of electron and particle with momentum \( p_1 - k \) and \( r_1 \) to momentum \( p_2 - k \) and \( r_2 \) and finally the emitted photon \( k \) is reabsorbed and we get to level 5 with outgoing particles with momentum \( p_2, r_2 \).

Let's calculate the scattering amplitude of \( p_1, r_1 \) to \( p_2, r_2 \) and in the process calculate the new transition amplitude of scattering from \( p_1 \) to \( p_2 \). This modification of amplitude of scattering from \( p_1 \) to \( p_2 \) as compared to one studied in section 3.2 is called the Vertex correction.

Observe under non-relativistic limit

\[ E_{12} = E_1 - E_2 = E_{p_1} - (E_{p_1-k} + E_k) \sim E_{p_2} - (E_{p_2-k} + E_k) = E_{45} \] (3.135)

Then the transition amplitude from level 1 to level 5 is a third order term and simply (see the end of the section)
Fig. 3.13 Fig. A shows moller scattering of incoming electron with momentum $p_1$ and a heavy particle with momentum $p_2$. Incoming electron emits a photon with momentum $k$ that recombines with outgoing electron with momentum $p_2 - k$. Fig. B shows a equivalent five level system. The incoming particles with momentum $p_1, r_1$ are at level 1, Emission of a photon with momentum $k$ transits to level 2. Level 2, 3, 4 represent the Moller scattering and finally the emitted photon $k$ is reabsorbed and we get to level 5.

\[ \mathcal{M} = \frac{\Omega_1 \Omega_2 \Omega_3}{E_{12}}, \]  
where $\Omega_i$ are as in Fig. 3.13B.

\[ \mathcal{M} = \frac{\Omega_1 \Omega_2 \Omega_3}{(E_{p_1 - k} + E_k)^2 - E_{p_1}^2(1 - E_{p_1} + E_k)^2 - E_{p_1}^2}, \]  
where

\[ \Omega_1 = \frac{C}{\sqrt{2E_k}} \bar{u}(p_1 - k) \gamma^\nu \epsilon^*_\nu(k) u(p_1), \]  
\[ \Omega_2 \propto C^2 \frac{\bar{u}(r_2) \gamma^\mu \epsilon_\mu(q) u(r_1) \bar{u}(p_2 - k) \gamma^\mu \epsilon^*_\mu(q) u(p_1 - k)}{q^2}, \]  
\[ \Omega_3 = \frac{C}{\sqrt{2E_k}} \bar{u}(p_2) \gamma^\nu \epsilon(k) u(p_2 - k). \]
Define CM frame as \( p_1 - k = 0 \). Then we find, in CM frame, \( p_1 \) and \( k \) are relativistic, \( \frac{E_{p_1}}{E_k} \sim 1 \) and \( \frac{E_{p_1}-k}{E_k} \sim 0 \), and \( (E_{p_1}-k+E_k)^2 - E_{p_1}^2 = m_0E_k \). Then in CM frame where \( E_{p_1}-k = m_0 \) can bound \( M \) as

\[
\mathcal{M} < E_b \propto \frac{C^4}{q^2} \frac{m_0^2}{(E_{p_1}-k+E_k)^2 - E_{p_1}^2E_{p_1}-k}
\]

(3.141)

\( E_b \) is Lorentz amplitude. In lab frame. Observe for large \( |k| \), the denominator in Eq. (3.141) goes as \( |k|^5 \) and we can sum over \( k \) to get a finite expression. After summing over \( k \), gives the transition amplitude of \( p_1 \) to \( p_2 \) and \( r_1 \) to \( r_2 \) and hence the vertex correction.

We end the section by sketching the proof for Eq. 3.136. The state of the four level system (level \( 1, 2, 4, 5 \) in Fig. 3.13B) evolves according to the Schrödinger equation

\[
\dot{\psi} = -\frac{i}{\hbar} \begin{bmatrix}
E_1 & \Omega_1^* & 0 & 0 \\
\Omega_1 & E_2 & \Omega_2^* & 0 \\
0 & \Omega_2 & E_3 & \Omega_3^* \\
0 & 0 & \Omega_3 & E_1
\end{bmatrix} \psi.
\]

(3.142)

We proceed into the interaction frame of the natural Hamiltonian (system energies) by transformation

\[
\phi = \exp\left(\frac{i}{\hbar} \frac{\Omega_1 t}{1} \begin{bmatrix}
E_1 & 0 & 0 & 0 \\
0 & E_2 & 0 & 0 \\
0 & 0 & E_3 & 0 \\
0 & 0 & 0 & E_1
\end{bmatrix}\right) \psi.
\]

(3.143)

This gives for \( E_{12} = E_2 - E_1 \),

\[
\dot{\phi} = -\frac{i}{\hbar} \begin{bmatrix}
0 & \exp\left(-\frac{i}{\hbar} E_{12} t\right) & 0 & 0 \\
\exp\left(\frac{i}{\hbar} E_{12} t\right) & 0 & \Omega_1^* & 0 \\
0 & \Omega_2 & 0 & \exp\left(\frac{i}{\hbar} E_{12} t\right) & \Omega_2^* \\
0 & 0 & \exp\left(-\frac{i}{\hbar} E_{12} t\right) & 0
\end{bmatrix} \phi.
\]

(3.144)

\( H(t) \) is periodic with period \( \Delta t = \frac{2\pi}{E_{12}} \). After \( \Delta t \), the system evolution is \( \phi(\Delta t) = \)

\[
(I + \int_0^{\Delta t} H(\sigma)d\sigma + \int_0^{\Delta t}\int_0^{\sigma_1} H(\sigma_1)H(\sigma_2)d\sigma_2d\sigma_1 + \int_0^{\Delta t}\int_0^{\sigma_2} H(\sigma_1)H(\sigma_2)H(\sigma_3)d\sigma_3d\sigma_2d\sigma_1 \ldots )\phi(0).
\]

(3.145)

The first integral averages to zero, while the second integral doesn’t give transition between 1 and 4. The third order does with a contribution.
\[
\int_0^{\Delta t} \Omega_1 \exp\left(\frac{i}{\hbar} E_{12} \sigma_1 \right) \int_0^{\sigma_1} \Omega_2 \int_0^{\sigma_2} \Omega_3 \exp\left(-\frac{i}{\hbar} E_{12} \sigma_3 \right) d\sigma_3 d\sigma_2 d\sigma_1 = 2\Delta t \frac{\Omega_1 \Omega_2 \Omega_3}{E_{12}^2}.
\]
Chapter 4
Weak Interactions

4.1 Massive Fields

We equip massive $A$ with a dynamics by defining Lagrangian as density

$$L = \varepsilon_0 \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} \left( \frac{mc}{\hbar} \right)^2 A_\mu A^\mu \right). \tag{4.1}$$

We just write

$$L = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} m^2 A_\mu A^\mu. \tag{4.2}$$

where recall

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.3}$$

The energy density of this field is

$$H = -F_{0 \mu} F^{0 \mu} + \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} m^2 A_\mu A^\mu. \tag{4.4}$$

Variation of $L$ gives

$$\partial_\mu F^{\mu \nu} + m^2 A^\nu = 0 \tag{4.5}$$

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) + m^2 A^\nu = 0 \tag{4.6}$$

Observe

$$\partial_\mu F^{\mu \nu} = 0 \tag{4.7}$$

which gives

$$\partial_\mu A^\mu = 0 \tag{4.8}$$
\[
\partial_\mu \partial^\mu A^\nu + m^2 A^\nu = 0 \quad (4.9)
\]

or
\[
\left( \frac{\partial^2}{c^2 dt^2} - \nabla^2 + m^2 \right) A^\nu = 0 \quad (4.10)
\]

Solution is \( \varepsilon \exp(j(kx - \omega t)) \), where
\[
k_0 = \frac{\omega}{c} = \sqrt{k^2 + m^2} \quad (4.11)
\]

\[
\partial_\mu A^\mu = 0 \rightarrow k \varepsilon^\mu = 0 \quad (4.12)
\]

Consider field in \( z \) direction. There are three independent polarization directions
\[
\varepsilon_1 = (0, 1, 0, 0) \quad (4.13)
\]
\[
\varepsilon_2 = (0, 0, 1, 0) \quad (4.14)
\]
\[
\varepsilon_3 = \frac{1}{m}(k, 0, 0, k_0) \quad (4.15)
\]

For example, consider a massive photon
\[
A \varepsilon_{1,2} \cos(k \cdot z - \omega t), \quad (4.16)
\]

propagating in \( z \) direction with \( \frac{\omega}{c} = \sqrt{k^2 + m^2} \). From 4.4, the energy of this photon is \( \varepsilon_0 \frac{k_0^2}{c^2} V \). Therefore for \( \varepsilon_0 \frac{k_0^2}{c^2} V = \hbar \omega \), we have the photon
\[
A = c \sqrt{\frac{2\hbar}{V \varepsilon_0 \omega}} \varepsilon_{1,2} \cos(k \cdot z - \omega t) = c \sqrt{\frac{\hbar}{2 \varepsilon_0 \omega V}} \varepsilon_{1,2} \left( \exp i(k \cdot z - \omega t) + \exp -i(k \cdot z - \omega t) \right).
\]

Consider the massive photon
\[
A \varepsilon_3 \cos(k \cdot z - \omega t), \quad (4.18)
\]

propagating in \( z \) direction. The energy of this photon is \( \varepsilon_0 \frac{k_0^2 m^2}{2} \). Therefore for \( \varepsilon_0 \frac{k_0^2 m^2}{2} V = \hbar \omega \), we have the photon
\[
A = \sqrt{\frac{2\hbar \omega}{V \varepsilon_0 m^2}} \varepsilon_3 \cos(k \cdot z - \omega t) \sim c \sqrt{\frac{\hbar}{2 \varepsilon_0 \omega}} \varepsilon_3 \cos(k \cdot z - \omega t). \quad (4.19)
\]

where last approximation true when \( k << m \).
4.2 Charged Weak Interaction

There are two charged massive bosons that mediate charged weak interaction. The Boson $W^+$ with momentum $k$ takes in an electron of momentum $p$ and emits a neutrino of momentum $p+k$ as shown in Fig. 4.1A. The amplitude for the transition is

$$\Omega = \frac{C}{\sqrt{2m}} \bar{u}_2(p+k)\tilde{\psi}^{\nu}(k)\gamma^\nu u_1(p),$$  \hspace{1cm} (4.20)

where $C = \frac{h \alpha_e}{\sqrt{2}}$. Here $g_w$ is weak coupling constant and analogous to $\frac{e}{\sqrt{4\pi}}$ in QED, and

$$\tilde{\psi}^{\nu} = \gamma^\nu \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \gamma^\nu \frac{1 - \gamma^5}{2}$$  \hspace{1cm} (4.21)

where $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. This ensures only left $\psi_L$ of the spinor $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ takes part in weak interaction. This is called a V-A vertex of weak interaction and arises from parity violation in weak interaction as explained subsequently.

In Fig. 4.1C, the Boson $W^+$ with momentum $k$ emits a positron of momentum $-p$ and emits a neutrino of momentum $p+k$ as shown in Fig. 4.1A. The amplitude for the transition is
\[ \Omega = \frac{C}{\sqrt{2m}} \bar{u}_2(p+k) \gamma^\nu \epsilon_\nu(k) u_1(p). \]  
(4.22)

In Fig. 4.1B we consider Boson $W^-$ instead of $W^+$. The Boson $W^-$ with momentum $k$ takes in a neutrino of momentum $p$ and emits a neutrino of momentum $p+k$ as shown in Fig. 4.1A. The amplitude for the transition is

\[ \Omega = \frac{C}{\sqrt{2m}} \bar{u}_2(p+k) \gamma^\nu \epsilon_\nu(k) u_1(p). \]  
(4.23)

### 4.3 Inverse Muon Decay

Consider the following process mediated by weak force.

\[ e + \nu_\mu \rightarrow \nu_e + \mu \]  
(4.24)

Electron and muon neutrino with momentum $p_1$ and $p_2$ collide to produce electron neutrino and muon at momentum $p_3$ and $p_4$. Let $k$ and $q$ denote the on-shell and off-shell momenta of mediator $W$ boson.

With

\[ \Omega_1 = \frac{C}{\sqrt{2m}} \bar{u}(p_3) \gamma^\nu \epsilon_\nu(k) u(p_1) \]  
(4.25)

\[ \Omega_2 = \frac{C}{\sqrt{2m}} \bar{u}(p_4) \gamma^\mu \epsilon_\mu(k) u(p_2). \]  
(4.26)
4.3 Inverse Muon Decay

The amplitude for the process

\[
\mathcal{M} = \Omega_1 \Omega_2 \left( \frac{1}{E_e(p_1) + E_{\nu_e}(p_3) - E_{W^-}(q)} - \frac{1}{E_{W^+}(-q) + E_{\mu^-}(p_4) - E_{\nu_e}(p_2)} \right)
\]

\[
= \Omega_1 \Omega_2 \left( \frac{1}{E_e(p_1) + E_{\nu_e}(p_3) - E_{W^-}(q)} - \frac{1}{E_{W^+}(-q) + E_e(p_1) - E_{\nu_e}(p_3)} \right)
\]

\[
= \Omega_1 \Omega_2 \left( \frac{2E_{W^-}(q)}{q^2 - m_W^2} \right)
\]

\[
\sim \frac{C^2}{q^2 - m_W^2} \bar{u}(p_4) \gamma^\mu \gamma^\nu \gamma^\kappa \epsilon(k) u(p_2) \bar{u}(p_3) \gamma^\kappa \gamma^\nu \gamma^\mu \epsilon(k) u(p_1)
\]

Now we have to sum over the polarization \( \epsilon \).

### 4.3.1 Polarization sum

With \( W^- \) say along the \( z \) direction, we have three polarization of the Boson

\[
\epsilon_1 = (0, 1, 0, 0) \quad (4.27)
\]

\[
\epsilon_2 = (0, 0, 1, 0) \quad (4.28)
\]

\[
\epsilon_3 = \frac{1}{m_W} (k, 0, 0, k_0) \quad (4.29)
\]

We sum over the polarization to find for

\[
x = \begin{bmatrix} \bar{u}(p_3) \gamma^\mu u(p_1) \\ \bar{u}(p_3) \gamma^\mu u(p_1) \\ \bar{u}(p_3) \gamma^\mu u(p_1) \end{bmatrix}, \quad y = \begin{bmatrix} \bar{u}(p_4) \gamma^\mu u(p_2) \\ \bar{u}(p_4) \gamma^\mu u(p_2) \\ \bar{u}(p_4) \gamma^\mu u(p_2) \end{bmatrix}
\]

\[
\Omega_1 \Omega_2 \sim y' \sum \epsilon_i \epsilon'_i x
\]

\[
\sum_i \epsilon_i \epsilon'_i =
\]

\[
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{m_W^2} \begin{pmatrix} k_0^2 & 0 & 0 & k_0 \\ 0 & 0 & 0 & 0 \\ k_0 & 0 & 0 & k_0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{m_W^2} \begin{pmatrix} k_0^2 & 0 & 0 & k_0 \\ 0 & 0 & 0 & 0 \\ k_0 & 0 & 0 & k_0 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{kk'}{m_W^2}
\]

Under Lorentz boosts \( g_{\mu,\nu} \) is invariant but the part \( kk' \) is not because boson \( W^- \) transforms as off-shell four momentum \( q \) not as \( k \) under lorentz boosts. To make
Weak Interactions

The weak interactions are Lorentz invariant, we make a leap, we just agree $\frac{kk'}{m_W^2}$ is not there so that everything is Lorentz invariant. This gives With $q^2 \ll m_W^2$, we then get

$$\mathcal{M} \sim C^2 \frac{\bar{u}(p_4) \gamma^\mu (1-\gamma^5) u(p_2)}{m_W^2} \bar{u}(p_3) \gamma_\mu u(p_1)$$

(4.32)

$$= C^2 \frac{\bar{u}(p_4) \gamma^\mu (1-\gamma^5) u(p_2)}{4m_W^2} \bar{u}(p_3) \gamma_\mu u(p_1).$$

(4.33)

$$\sum_s |\mathcal{M}|^2 = \left( \frac{C^2}{16m_W^2} \right)^2 \frac{1}{E_1E_2E_3E_4} \text{Tr}(p_3 \gamma^\mu (1-\gamma^5)(p_1 + m_e) \gamma^\nu (1-\gamma^5)) \times \text{Tr}((p_4 + m_\mu) \gamma_\mu (1-\gamma^5) p_2 \gamma_\nu (1-\gamma^5)).$$

(4.34)

With lots of algebra,

$$\sum_s |\mathcal{M}|^2 = \left( \frac{C^2}{m_W^2} \right)^2 \frac{(p_1 \cdot p_2) (p_3 \cdot p_4)}{E_1E_2E_3E_4}$$

(4.35)

In The CM frame where we neglect electron mass, we find, $E_1 = E_2 = E$ and $E_3 \sim E_4 = E$ and $E_3^2 - E_2^2 = m_\mu^2$. $E \sim \sqrt{|p|^2 + m_\mu^2/2}$, where $p$ is the Muon momentum.

$$\sum_s |\mathcal{M}|^2 = \left( \frac{2C^2}{m_W^2} \right)^2 (1 - \frac{m_\mu^2}{2E^2})$$

(4.36)

$$\frac{d\sigma}{d\Omega} = \left( \frac{2\alpha_w}{m_W^2} \right)^2 \frac{\hbar c}{E} \left( 1 - \frac{m_\mu^2}{2E^2} \right)^2$$

(4.37)

4.4 Muon Decay

Fig. 4.4A shows the decay of a muon where the amplitude of the Feynman diagram is In Fig. 4.4B

$$\Omega_1 = \frac{C}{\sqrt{2m_W^2}} \bar{u}(p_3) \gamma^\nu \epsilon^* \epsilon(k) u(p_1)$$

(4.38)

$$\Omega_2 = \frac{C}{\sqrt{2m_W^2}} \bar{u}(p_4) \gamma^\mu \epsilon \epsilon(k) v(p_2).$$

(4.39)
where last equality follows after polarization sum.

\[
\sum_s |\mathcal{M}|^2 = \left(\frac{C^2}{16m^2_W}\right)^2 \frac{1}{E_1 E_2 E_3 E_4} \text{Tr}(p_3 \gamma^\mu (1 - \gamma^5)(p_1 + m_\mu) \gamma^\nu (1 - \gamma^5)) \times \text{Tr}((p_4 + m_e) \gamma_\mu (1 - \gamma^5)p_2 \gamma_\nu (1 - \gamma^5)).
\] (4.40)

With lot of algebra,

\[
\sum_s |\mathcal{M}|^2 = \left(\frac{C^2}{m^2_W}\right)^2 \frac{(p_1 \cdot p_2)(p_3 \cdot p_4)}{E_1 E_2 E_3 E_4}
\] (4.41)
\[ p_1 \cdot p_2 = E_1 E_2 \]  
\[ p_3 \cdot p_4 = E_3 E_4 \left( 1 + \frac{l \cos \theta + k}{\sqrt{\frac{k^2}{4} + l^2 + kl \cos \theta}} \right) \]  
(4.42)  
(4.43)

Writing
\[ \frac{(hc)^{6}}{V^2} = \frac{k^2 \Delta k}{(2\pi)^3} \frac{l^2 \Delta l}{(2\pi)^3} d\Omega_1 d\Omega_2 \]  
(4.44)

Integrating above over \( \Omega_1 \) and \( \Omega_2 \) we get
\[ \frac{l}{r} = \tan \theta_1 \]
\[ \Sigma(\theta_1) = r^5 \cos^2 \theta_1 \sin^2 \theta_1 \Delta r \Delta \theta_1 \]  
(4.45)

\[ \frac{(p_1 \cdot p_2) (p_3 \cdot p_4)}{E_1 E_2 E_3 E_4} = \left( 1 + \frac{\tan \theta_1 \cos \theta + \frac{1}{2}}{\sqrt{\frac{k^2}{4} + \tan^2 \theta_1 + \tan \theta_1 \cos \theta}} \right) = g(\theta, \theta_1). \]  
(4.46)

\[ m_\mu = E = E_2 + E_3 + E_4 = k + \sqrt{\frac{k^2}{4} + l^2 + kl \cos \theta + \sqrt{\frac{k^2}{4} + l^2 - kl \cos \theta}} = rf(\theta, \theta_1). \]  
(4.47)

\[ \Delta E = \frac{\partial E}{\partial r} \left( 1 + \left( \frac{\partial f}{\partial \theta} \right)^2 \right) \Delta r = f(1 + \left( \frac{\partial f}{\partial \theta} \right)^2) = h(\theta, \theta_1) \]  
(4.48)

Decay rate \( \Gamma = \)

\[ \pi \frac{\left| \Sigma \right|^2}{\Delta E} = \frac{2}{\pi} \frac{\alpha^2}{m_W^2} \int \int \Sigma(\theta_1) \frac{g(\theta, \theta_1)}{h(\theta, \theta_1)} \, d\theta_1 \, d\theta. \]  
(4.49)

\[ = A \frac{\alpha^2}{m_W^2} \frac{m_\mu^5}{m_W^4}. \]  
(4.50)

\[ A = \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^2 \theta_1 \sin^2 \theta_1 \frac{g(\theta, \theta_1)}{f^3(\theta, \theta_1) h(\theta, \theta_1)} \, d\theta_1 \, d\theta \]

4.5 Pion Decay

Consider charged Pion decay as shown in Fig. (4.5).
\[ \pi^- \rightarrow \pi_0 + e + \bar{\nu}_e \]  

(4.51)

It is same a muon decay except now instead of emitting a muon neutrino we emit a neutral pion. However the amplitude of the process is same as in Eq. (4.41)

\[ \sum_s |\mathcal{M}|^2 = \left( \frac{C^2}{m_W^2} \right)^2 \frac{(p_1 \cdot p_2)(p_3 \cdot p_4)}{E_1E_2E_3E_4} \]  

(4.52)

\[ p_1 \cdot p_2 = E_1E_2 \]  

(4.53)

\[ p_3 \cdot p_4 = E_3E_4(1 + k \frac{l \cos \theta + \frac{k}{r}}{\sqrt{m^2_\pi + l^2 + kl \cos \theta} \sqrt{m^2_\pi_0 + k^2}}) \]  

(4.54)

\[ \sim E_3E_4(1 + \frac{k}{m_\pi_0} \frac{l \cos \theta + \frac{k}{r}}{\sqrt{l^2 + l^2 + kl \cos \theta}}) \]  

(4.55)

Writing

\[ \frac{(hc)^6}{V^2} = \frac{k^2 \Delta k}{(2\pi)^3} \frac{l^2 \Delta l}{(2\pi)^3} d\Omega_1 d\Omega_2 \]  

(4.56)

Integrating above over \( \Omega_1 \) and \( \Omega_2 \) we get \( \frac{l}{k} \Delta k \Delta l d\theta \). Let \( r = \sqrt{k^2 + l^2} \) and

\[ \frac{l}{k} = \tan \theta_i \]
\[ \Sigma(\theta_1) = r^5 \cos^2 \theta_1 \sin^2 \theta_1 \Delta r \Delta \theta_1 \]  
(4.57)

\[ \frac{(p_1 \cdot p_2)}{E_1} \frac{(p_3 \cdot p_4)}{E_2 E_3 E_4} = (1 + \frac{r \cos \theta_1}{m_{\pi_0}} \tan \theta_1 \cos \theta + \frac{1}{2}) \frac{1}{\sqrt{1 + \tan^2 \theta_1 + \tan \theta_1 \cos \theta}} = (1 + \frac{r}{m_{\pi_0}} g(\theta, \theta_1)). \]  
(4.58)

\[ m_{\pi^-} = E = E_2 + E_3 + E_4 = m_{\pi_0} + \sqrt{\frac{k^2}{4} + l^2 + kl \cos \theta} + \sqrt{\frac{k^2}{4} + l^2 - kl \cos \theta} = m_{\pi_0} + rf(\theta, \theta_1). \]  
(4.59)

\[ \Delta E = \frac{\partial E}{\partial r}(1 + \left( \frac{\partial E}{\partial \theta} \frac{\partial \theta}{\partial r} \right)^2) \Delta r = f(1 + \left( \frac{\partial f}{\partial \theta} \right)^2) = h(\theta, \theta_1) \]  
(4.60)

Decay rate \( \Gamma = \)

\[ \pi \int \frac{|\mathcal{M}|^2}{\Delta E} = \frac{2}{\pi} \frac{\alpha_e^2}{m_W} \int \frac{1}{\Delta E} \int \Sigma(\theta_1) \frac{1 + \frac{r}{m_{\pi_0}} g(\theta, \theta_1)}{h(\theta, \theta_1)} d\theta_1 d\theta. \]  
(4.61)

\[ = A \frac{\alpha_e^2}{m_W} \frac{(m_{\pi^-} - m_{\pi_0})^5}{m_W^4}. \]  
(4.62)

\[ A = \frac{2}{\pi} \int_0^\pi \int_0^{\pi} \frac{1}{2} \cos^2 \theta_1 \sin^2 \theta_1 \frac{1}{f^5(\theta, \theta_1) h(\theta, \theta_1)} d\theta_1 d\theta. \]

4.6 More Pion Decay

\[ \pi^- \to e + \bar{\nu}_e \]  
(4.63)

\[ d + \bar{u} \to e + \bar{\nu}_e \]  
(4.64)

\( \pi^- \) is a bound state of d quark and u antiquark. The bound state can be written as sum of states like

\[ \phi = \exp(ik \cdot \frac{(r_1 + r_2)}{2}) \exp(il \cdot \frac{(r_1 - r_2)}{2}) = \exp(ip_1 r_1) \exp(-ip_3 r_2) \]

with different \( l' \)'s as shown in 4.8A corresponding to different \( \theta_1 \). Then \( p_1 = k/2 + l \) and \( -p_3 = k/2 - l \) as in 4.8B. For pion at rest \( k = 0 \). The energy of the pion then is
\[ m_\pi = E = \sqrt{m_u^2 + p_1^2} + \sqrt{m_d^2 + p_2^2} = \sqrt{m_u^2 + \frac{k^2}{4} + l^2 + kl \cos \theta_1} + \sqrt{m_d^2 + \frac{k^2}{4} + l^2 - kl \cos \theta_1} \sim 2l \quad (4.65) \]

when \( k = 0 \).

Let us calculate the decay rate for one configuration \( \theta_1 = 0 \). The total decay rate then is the average over \( \theta_1 \) which by symmetry is just as for \( \theta_1 = 0 \).

\[ \mathcal{M} \sim \frac{C^2}{m_W^2} \bar{u}(p_4) \gamma^\mu u(p_2) \bar{v}(p_3) \gamma_\mu u(p_1) \quad (4.66) \]

\[ = \frac{C^2}{4m_W^2} \bar{u}(p_4) \gamma^\mu (1 - \gamma^5) u(p_2) \bar{v}(p_3) \gamma_\mu (1 - \gamma^5) u(p_1). \quad (4.67) \]

\[ \sum_s |\mathcal{M}|^2 = \left( \frac{C^2}{16m_W^2} \right)^2 \frac{1}{E_1E_2E_3E_4} Tr(p_1 \gamma^\mu (1 - \gamma^5) (p_1 + m_\mu) \gamma^\nu (1 - \gamma^5)) \times Tr((p_4 + m_\mu) \gamma_\mu (1 - \gamma^5) p_2 \gamma_\nu (1 - \gamma^5)). \quad (4.68) \]

With lot of algebra,
\[ \sum_s |\mathcal{M}|^2 = \left( \frac{C^2}{m_W} \right)^2 \frac{(p_1 \cdot p_2)(p_3 \cdot p_4)}{E_1E_2E_3E_4} \]  
\[ = \left( \frac{C^2}{m_W} \right)^2 (1 - \cos^2 \theta) \]  
\[ = \left( \frac{C^2}{m_W} \right)^2 \sin^2 \theta \]  
(4.69)  
(4.70)  
(4.71)

\[ m_\pi = E = E_3 + E_4 = \sqrt{m_e^2 + \frac{k^2}{4} + l^2 + kl \cos \theta + \sqrt{\frac{k^2}{4} + l^2 - kl \cos \theta}} \sim 2l. \]  
(4.72)

\[ \Delta E = 2\Delta l \]  
(4.73)

\[ \frac{(\hbar c)^3}{V^3} = \frac{l^2 \Delta l}{(2\pi)^3} d\Omega_1 \]  
(4.74)

Decay rate \( \Gamma = \)

\[ \frac{\pi}{\Delta E} \sum_s |\mathcal{M}|^2 = \frac{\pi}{2} \frac{\alpha^2 m^2_\pi}{m_W} \frac{(\hbar c)^3}{V_0} \int \sin^2 \theta d\theta. \]  
\[ = \frac{\pi}{2} \frac{(\hbar c)^3}{V_0} \frac{\alpha^2 m^2_\pi}{m_W} \]  
(4.75)  
(4.76)

With \( V_0 \) corresponding to pion radius of 1 fm. We find \( \frac{(\hbar c)^3}{V_0} \sim 200(Mev)^3 \).

### 4.7 Neutral Weak Interactions

#### 4.7.1 Elastic Neutrino-electron scattering

\[ \nu_\mu + e \overset{Z}{\rightarrow} \nu_\mu + e \]  
(4.77)

\[ \mathcal{M} \sim \frac{C^2}{4m_Z^2} \bar{u}(p_4) \gamma^\mu (c_V - c_A \gamma^5) u(p_2) \bar{u}(p_3) \gamma_\mu (1 - \gamma^5) u(p_1). \]  
(4.78)

\[ \sum_s |\mathcal{M}|^2 = \left( \frac{C^2}{16m_Z^2} \right)^2 \frac{1}{E_1E_2E_3E_4} \text{Tr}((\gamma_3 + m_e) \gamma^\mu (c_V - c_A \gamma^5)(\gamma_3 + m_e) \gamma^\nu (c_V - c_A \gamma^5)) \times \text{Tr}(\gamma_4 \gamma_\mu (1 - \gamma^5) \gamma_5 \gamma_\nu (1 - \gamma^5)). \]
With lot of algebra, and $E$ as CM energy

$$
\sum_s |\mathcal{M}|^2 = \left( \frac{C^2}{m_Z^2} \right)^2 \left( \frac{(c_A + c_V)^2 (p_1 \cdot p_2) (p_3 \cdot p_4) + ((e_A - c_V)^2 (p_1 \cdot p_4) (p_3 \cdot p_3) + m_e^2 (c_A^2 - c_V^2) (p_1 \cdot p_3)}{E_1 E_2 E_3 E_4} \right)
$$

$$
\frac{d\sigma}{d\theta} = 4\pi \left( \frac{\alpha \hbar c E}{m_Z^2} \right)^2 \left( (c_A + c_V)^2 + ((e_A - c_V)^2 \cos^4 \frac{\theta}{2}) \right)
$$

### 4.7.2 Electron Positron scattering

$$
\mathcal{M} = \frac{C^2}{4(q^2 - m_Z^2)} \bar{u}(p_4) \gamma^\mu (e'_V - c'_A \gamma^5) v(p_3) \bar{v}(p_2) \gamma_\mu (e'_V - c'_A \gamma^5) u(p_1). (4.79)
$$
\[
\sum_s |\mathcal{M}|^2 = \frac{C^2}{16(q_z^2 - m_Z^2)^2} \frac{1}{E_1 E_2 E_3 E_4} Tr(\gamma^\mu (c^f_\nu - c^f_A \gamma^5) \gamma^\nu (c^f_\nu - c^f_A \gamma^5)) \\
\times Tr(\gamma^\mu (c^e_\nu - c^e_A \gamma^5) \gamma^\nu (c^e_\nu - c^e_A \gamma^5)).
\]

\[
\sum_s |\mathcal{M}|^2 = \frac{1}{2 (q_z^2 - m_Z^2)} \frac{C^2}{E_1 E_2 E_3 E_4} \left\{ (c^f_A)^2 + (c^f_\nu)^2 \right\} \left\{ (c^e_A)^2 + (c^e_\nu)^2 \right\} \left\{ (p_4 \cdot p_2) (p_3 \cdot p_4) + (p_1 \cdot p_4) (p_2 \cdot p_3) \right\} \\
+ 4 c^e_\nu c^f_\nu c^f_A [(p_1 \cdot p_2) (p_3 \cdot p_4) - (p_1 \cdot p_4) (p_3 \cdot p_2)]
\]

In CM frame it reduces to

\[
\sum_s |\mathcal{M}|^2 = \frac{C^2}{2 (2E)^2 - m_Z^2} \frac{1}{E_1 E_2 E_3 E_4} \left\{ (c^e_A)^2 + (c^e_\nu)^2 \right\} \left\{ (c^f_A)^2 + (c^f_\nu)^2 \right\} \left[ 1 + \cos^2 \theta \right] \\
- 8 c^e_\nu c^f_\nu c^f_A c^e_A \cos \theta
\]

The differential cross-section

\[
\sum_s |\mathcal{M}|^2 = \frac{C^2}{2 (2E)^2 - m_Z^2} \left\{ (c^e_A)^2 + (c^e_\nu)^2 \right\} \left\{ (c^f_A)^2 + (c^f_\nu)^2 \right\} \left[ 1 + \cos^2 \theta \right] \\
- 8 c^e_\nu c^f_\nu c^f_A c^e_A \cos \theta
\]
The differential cross section is

\[
\frac{d\sigma}{d\theta} = \pi \left( \frac{\alpha_w \hbar c E}{(2E)^2 - m_Z^2} \right)^2 \left\{ (c_A^f)^2 + (c_V^f)^2 \right\} \left\{ (c_A^f)^2 + (c_V^f)^2 \right\} \left[ (1 + \cos^2 \theta) \right] - 8c_V^f c_A^f c_V^f c_A^f \cos \theta \right\}
\]

4.8 Electroweak Unification, Parity violation and mass

4.8.1 Introduction

Beginning with the seminal work of Yang and Lee [1] and its experimental verification by Wu [2], it is well known that weak interactions do not preserve parity. In the theory of weak interactions, this is manifested by coupling only the left handed components of the fermion doublet. The work of parity violation began with Yang and Lee’s observations on K-mesons which led them to question parity conservation in weak interactions. This led them to devise many experiments that would test parity conservation. The first of these was carried out by Wu [2], which confirmed parity violation in weak interactions. Further developments in the theory of weak interactions include invention of Higg’s mechanism which gives masses to vector bosons and fermions [3, 4, 5] and the theory of electroweak unification [6, 7]. Historical facts suggest that work on parity violation preceded the work on Higg’s mechanism and electroweak unification. In this paper, we take a different viewpoint. We suggest that parity violation in weak interactions can be predicted on pure theoretical grounds. In this paper, we show that parity violation is a natural consequence of gauge invariance. In a theory where there is no parity violation, we cannot assign masses to fermions in a gauge invariant way using the Higg’s mechanism because Higg’s field transforms in a quadratic way under gauge transformation. However when we violate parity and only couple the left handed components of the fermions, Higg’s field transforms in a linear way under gauge transformation and it becomes possible to give masses to fermions in a gauge invariant manner.

The paper is organized as follows. We first review the basics of Higg’s mechanism for giving masses to vector bosons and fermions [8, 9, 10]. We then go through the exercise of showing how the theory is gauge invariant, when we have parity violation. Then we work through a theory where there is no parity violation and show we cannot assign masses to fermions in a gauge invariant way using the Higg’s mechanism.

4.8.2 Theory

We consider the Higg’s doublet
\[ \Phi = \begin{bmatrix} \Phi_A \\ \Phi_B \end{bmatrix}. \] (4.80)

The field is coupled to electromagnetic field and W,Z bosons with gauge coupling, with Lagrangian density

\[ \mathcal{L}_\Phi = D_\mu \Phi^\dagger D^\mu \Phi - V(\Phi^\dagger \Phi), \] (4.81)

where

\[ D_\mu = \partial_\mu + i \frac{g_1}{2} B_\mu + i \frac{g_2}{2} W_\mu, \] (4.82)

with \( B_\mu \) and \( W_\mu \) the vector potential for EM and Weak interactions respectively and \( g_1 \) and \( g_2 \) as the corresponding coupling constants and

\[ V(\Phi^\dagger \Phi) = \frac{m^2}{2\phi_0^2} [(\Phi^\dagger \Phi) - \phi_0^2], \] (4.83)

where the ground state of the Higgs field is

\[ \Phi_{\text{ground}} = \begin{bmatrix} 0 \\ \phi_0 \end{bmatrix}, \] (4.84)

and the excited state

\[ \Phi = \begin{bmatrix} 0 \\ \phi_0 + \frac{h(x)}{\sqrt{2}} \end{bmatrix}. \] (4.85)

Substituting \( \Phi \) in Eq. (4.81) gives masses to \( W, Z \) bosons and the Higgs boson via gauge coupling.

\[ D_\mu \Phi = \begin{pmatrix} 0 \\ \frac{\partial_x h}{\sqrt{2}} \end{pmatrix} + i \frac{g_1}{2} \begin{pmatrix} 0 \\ B_\mu (\phi_0 + \frac{h(x)}{\sqrt{2}}) \end{pmatrix} + i \frac{g_2}{2} \begin{pmatrix} \sqrt{2} W_\mu^+ (\phi_0 + \frac{h(x)}{\sqrt{2}}) \\ -W_\mu^3 (\phi_0 + \frac{h(x)}{\sqrt{2}}) \end{pmatrix}, \] (4.86)

and

\[ \mathcal{L}_\Phi = \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{g_1^2}{4} (W_\mu^+ W^{\dagger + \mu} + W_\mu^- W^{\dagger - \mu}) (\phi_0 + \frac{h(x)}{\sqrt{2}})^2 + \frac{g_1^2 + g_2^2}{4} Z_\mu Z^{\mu} (\phi_0 + \frac{h(x)}{\sqrt{2}})^2 - V(h), \] (4.87)

where \( W_\mu^+ = \frac{W_\mu^1 - i W_\mu^2}{\sqrt{2}} \) and \( W_\mu^- = \frac{W_\mu^1 + i W_\mu^2}{\sqrt{2}} \) are the W bosons and

\[ Z_\mu = W_\mu^3 \cos \theta_w - B_\mu \sin \theta_w, \] (4.88)

the Z boson and the massless photon

\[ A_\mu = W_\mu^3 \sin \theta_w + B_\mu \cos \theta_w, \] (4.89)
where $\theta_w$ is the Weinberg angle

$$\cos \theta_w = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}, \quad \sin \theta_w = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}. \quad (4.90)$$

The field couples to fermions as follows. Let us consider the the neutrino-electron doublet written as a four vector

$$L = \begin{bmatrix} v_R \\ v_L \\ e_L \\ e_R \end{bmatrix}. \quad (4.91)$$

Using the notation $\sigma_j = (\sigma_x, \sigma_y, \sigma_z)$, $\sigma_0 = (\sigma_0, \sigma_x, \sigma_y, \sigma_z)$, and $\sigma_i = (\sigma_0, -\sigma_x, -\sigma_y, -\sigma_z)$, the doublet evolves ($\vec{n}$ and $c$ are implicit) as $i\frac{dL}{dt} =$

$$\begin{bmatrix}
  i\partial_j \sigma_j & m_e & 0 & 0 \\
  m_e & -i\partial_j \sigma_j + \frac{1}{2}(g_2 W^\mu_\mu - g_1 B_\mu)\sigma_\mu & \frac{g_2}{\sqrt{2}} W^\mu_\mu \sigma_\mu & 0 \\
  0 & \frac{g_2}{\sqrt{2}} W^\mu_\mu \sigma_\mu & -i\partial_j \sigma_j - \frac{1}{2}(g_2 W^\mu_\mu + g_1 B_\mu)\sigma_\mu & m_e \\
  0 & 0 & m_e & i\partial_j \sigma_j - g_1 B_\mu \overline{\sigma}_\mu \\
\end{bmatrix} \begin{bmatrix} v_R \\ v_L \\ e_L \\ e_R \end{bmatrix}. \quad (4.92)$$

where $m_e = c_e(\phi_0 + \frac{h(x)}{\sqrt{2}})$ and $m_v = c_v(\phi_0 + \frac{h(x)}{\sqrt{2}})$, with $c_e, c_v$ as coupling of electron and neutrino to Higg’s boson.

When we express the above equation in terms of the fields $Z_\mu, A_\mu$, it takes the form $i\frac{dL}{dt} =$

$$\begin{bmatrix}
  i\partial_j \sigma_j & m_e & 0 & 0 \\
  m_e & -i\partial_j \sigma_j + \frac{\cos \theta_w}{\sin \theta_w} Z_\mu \sigma_\mu - \frac{g_2}{\sqrt{2}} W^\mu_\mu \sigma_\mu & 0 & 0 \\
  0 & \frac{g_2}{\sqrt{2}} W^\mu_\mu \sigma_\mu & -i\partial_j \sigma_j - e(A_\mu + \cot \theta_w Z_\mu)\sigma_\mu & m_e \\
  0 & 0 & m_e & i\partial_j \sigma_j - e(A_\mu - \tan \theta_w Z_\mu) \overline{\sigma}_\mu \\
\end{bmatrix} \begin{bmatrix} v_R \\ v_L \\ e_L \\ e_R \end{bmatrix}. \quad (4.93)$$

where $g_1 \cos \theta_w = g_2 \sin \theta_w = e$, with $-e$, the electron charge.

Now we look at how equations (4.81) and (4.92) transform when we make a Gauge transformation on $W$ and $B$. The transformations are for $U \in SU(2)$, we have

$$W_\mu \rightarrow U(x)W_\mu U^\dagger(x) + \frac{i\partial_\mu U(x)U^\dagger(x)}{g_2/2}, \quad (4.94)$$

$$B_\mu \rightarrow B_\mu - \frac{\partial_\mu \theta(x)}{g_1/2}. \quad (4.95)$$

Then the Higg’s doublet transforms as...
\[ \Phi \rightarrow \Theta(x) \Phi \]  

(4.96)

where \( \Theta(x) = \exp(i\theta(x))U(x) \).

In terms of field \( \Phi \), the equation for \( L \) takes the form \( i \frac{dL}{dt} = \)

\[
\begin{bmatrix}
    i\partial_j \sigma_j \\
    c_v \Phi_A - i\partial_j \sigma_j + \frac{1}{2} (g_2 W^3_\mu - g_1 B_\mu) \sigma_\mu \\
    \frac{2}{\sqrt{2}} g_2 W^3_\mu \sigma_\mu \\
    0
\end{bmatrix}
\begin{bmatrix}
    v_L \\
    e_L \\
    c_v \Phi_B
\end{bmatrix}
\]

(4.97)

where under the gauge transformation \( L \) transforms as

\[
\begin{bmatrix}
    v_L \\
    e_L
\end{bmatrix}
\rightarrow \exp(-i\theta(x))U(x)
\begin{bmatrix}
    v_L \\
    e_L
\end{bmatrix}
\]

(4.98)

\[
e_R \rightarrow \exp(-i2\theta(x))e_R
\]

(4.99)

In equation (4.92) only \( e_L \) and \( v_L \) are coupled. \( e_R \) and \( v_R \) are not coupled. That is to say we have parity violation. We now show that this physical law is in fact a consequence of the fact that it is not possible to give masses to fermions in a manner that is gauge invariant (as above), if we donot violate parity.

To see this, let's reorganize the doublet as

\[
M = \begin{bmatrix}
    v \\
    e
\end{bmatrix}, \quad V = \begin{bmatrix}
    v_L \\
    v_R
\end{bmatrix}, \quad \beta = \begin{bmatrix}
    m_v \\
    m_e
\end{bmatrix}
\]

(4.100)

\[
i \frac{dM}{dt} = \begin{bmatrix}
    -i\partial_j \sigma_j + \frac{1}{2} (g_2 W^3_\mu - g_1 B_\mu) \sigma_\mu \\
    \frac{2}{\sqrt{2}} g_2 W^3_\mu \sigma_\mu \\
    -i\partial_j \sigma_j + \frac{1}{2} (g_2 W^3_\mu + g_1 B_\mu) \sigma_\mu
\end{bmatrix}
\begin{bmatrix}
    v_L \\
    e_L
\end{bmatrix}
\]

(4.101)

where, \( \beta = \sigma_5 \otimes \sigma_0 \) and \( \alpha_0 = \sigma_5 \otimes \sigma_0 \) (\( \sigma_0 \) is identity), with \( \sigma_\mu \) Pauli matrices. If we plan to write this Eq. (4.101), in terms of Higgs's field \( \Phi \), then we find that \( \Phi \) enters the term \( C \) above. To make it gauge invariant, this term should be of the form

\[
C(\Phi) = \Theta(x) \begin{bmatrix}
    m_v & 0 \\
    0 & m_e
\end{bmatrix} \Theta^\dagger(x)
\]

(4.102)

In the above, \( C(\Phi) \) cannot be expressed in terms of \( \Phi \) alone. The best we can write it as

\[
C(\Phi) = \begin{bmatrix}
    c_v \Phi_B^* & c_v \Phi_A \\
    -c_v \Phi_A^* & c_v \Phi_B
\end{bmatrix}
\begin{bmatrix}
    \exp(i\theta(x)) & 0 \\
    0 & \exp(-i\theta(x))
\end{bmatrix}
\]

(4.103)
which is still not just $\Phi$ dependent. Hence when $m_e \neq m_\nu$, we cannot make our equations gauge invariant unless we do a *parity violation*. Therefore parity violation arises as a consequence of gauge invariance.
Chapter 5
Bound States and Quantum Chromodynamics

5.1 Quantum Mechanics

5.1.1 Schrödinger Equation

In classical mechanics, we talk about a particle say an electron with a position \( x \) and velocity \( v \). In quantum mechanics, particle state is represented by complex waves \( \exp(ikx) \) or sum of such waves \( \sum_j \exp(ik_jx) \). In complex wave \( \exp(ikx) \), \( k \) is the wavenumber of the particle. The wave evolves in time as \( \exp(i(kx - \omega(k)t)) \), \( \omega(k) \) is the frequency of the wave and depends on wavenumber \( k \). The dependence \( \omega(k) \) is called the dispersion relation of the wave. First postulate of quantum mechanics is that the energy of the wave is \( E = \hbar \omega(k) \), where \( \hbar \) is a fundamental constant called Planck’s constant. Its units are angular momentum and in SI units its value is \( 6.6 \times 10^{-34} \).

The momentum of our complex wave \( \omega, k \) is simply \( \hbar k \).

Now from classical mechanics \( E = \frac{p^2}{2m} \). Then we get \( \hbar \omega = \frac{\hbar^2 k^2}{2m} \) or \( \omega = \frac{\hbar k^2}{2m} \). Thus my complex wave \( \psi(x,t) = \exp(i(kx - \omega(k)t)) \) satisfies

\[
\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}.
\]

This equation (5.1) is called Schrödinger equation. It is still true if we have

\[
\psi(x,t) = \sum_j \alpha_j \exp(ik_jx - \omega(k_j)t).
\]

as individual exponential satisfy these equation.

\( \psi(x,t) \) is called a wavefunction of electron, it is superposition of plane waves. This is a feature of quantum mechanics, we can be in superposition of states. It satisfies the Schrödinger equation. All we are saying is that if we start with initial state \( \psi(x) = \sum_j \alpha_j \exp(ik_jx) \), these ways will evolve by their characteristic ener-
gies as \( \psi(x,t) = \sum_j \alpha_j \exp(i(k_jx - \omega(k_j)t)) \) and \( \psi(x,t) \) satisfies the Schrödinger equation.

![Figure 5.1](image)

**Fig. 5.1** Figure shows how \( V(x) \) is decomposed as piecewise constant potential.

Now how does my wavefunction evolve if I have a potential \( V \). Then from classical mechanics \( E - V = \frac{p^2}{2m} \), implying \( \hbar \omega - V = \frac{\hbar^2 k^2}{2m} \) or my wave satisfies

\[
i\hbar \frac{\partial \psi}{\partial t} = (-\hbar \frac{\partial^2}{\partial x^2} + V)\psi. \tag{5.2}
\]

and again same is true if we have superposition of plane waves.

Now how does the evolution of \( \psi(x) \) take place when we have \( V(x) \). Then we can break \( \psi(x) \) into small pieces \( \phi_i \) over which \( V(x) \) is constant as \( V_i \). See fig 5.1. Then each \( \phi_i \) sees a potential \( V_i \). Its evolution will be same if \( V_i \) was globally true. Then we can break \( \phi \) into exponentials and conclude it satisfies the equation

\[
i\hbar \frac{\partial \phi_i}{\partial t} = (-\hbar \frac{\partial^2}{\partial x^2} + V_i)\phi_i. \tag{5.3}
\]

Then adding them all we get

\[
i\hbar \frac{\partial \psi}{\partial t} = (-\hbar \frac{\partial^2}{\partial x^2} + V(x))\psi. \tag{5.4}
\]

Thus we have derived a fundamental equation of quantum mechanics. Wavefunction \( \psi(x) \) has a probabilistic interpretation. \( \int_a^b |\psi(x)|^2 \, dx \) gives the probability of finding the particles in the interval \([a,b]\).

### 5.1.2 Hydrogen Atom

In polar coordinates \( r = \sqrt{x^2 + y^2} \) and \( \phi = \tan^{-1}(\frac{y}{x}) \).

Then
\[
\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \phi}{r} \frac{\partial}{\partial \phi},
\]
\[
\frac{\partial}{\partial y} = \frac{\partial r}{\partial y} + \frac{\partial \phi}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \phi}{r} \frac{\partial}{\partial \phi}.
\]
\[
\frac{\partial^2}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}.
\]
\[
\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}.
\]
\[
\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial z^2} + \frac{1}{R^2} \frac{\partial}{\partial R} \left( \frac{1}{R^2} \frac{\partial}{\partial R} + \frac{1}{R^2 \sin^2 \theta} \frac{\partial}{\partial \phi^2} \right)
\]
\[
\frac{1}{R^2} \frac{\partial}{\partial R} \left( \frac{1}{R^2} \frac{\partial}{\partial R} + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{R^2 \sin^2 \theta} \frac{\partial}{\partial \phi^2} \right).
\]

Using \( R = \sqrt{z^2 + r^2} \) and \( \theta = \tan^{-1}(\frac{z}{r}) \).

\[
\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial z^2} + \frac{1}{R^2} \frac{\partial}{\partial R} \left( \frac{1}{R^2} \frac{\partial}{\partial R} + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{R^2 \sin^2 \theta} \frac{\partial}{\partial \phi^2} \right)
\]
\[
+ \left( \frac{2mR^2}{\hbar^2} (E - V(R)) \right) \psi = 0.
\]

We write the solution \( \psi = f(R)Y(\theta, \phi) \).

\[
(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \phi^2} + \frac{2mR^2}{\hbar^2} (E - V(R))) \psi = 0.
\]

Writing \( Y(\theta, \phi) = \Theta(\theta) e^{im\phi} \), we get

\[
(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) - \frac{m^2}{\sin^2 \theta} \Theta(\theta) = 0.
\]

For \( x = \cos \theta \), the above equation reads

\[
(1 - x^2) \Theta'' - 2x \Theta' + (l(l+1) - \frac{m^2}{1-x^2}) \Theta = 0.
\]

The solution \( \Theta^m_l \) exits for integer \( l, m \) satisfying \( 0 \leq |m| \leq l \). For \( m \geq 0 \)
\[ \Theta_l^m(x) = \frac{(-1)^m}{2^{1/2}} \frac{d^{l+m}}{dx^{l+m}}(x^2 - 1)^l. \]

with

\[ \Theta_l^{-m}(x) = \frac{(-1)^m}{(l+m)!} \Theta_l^m(x). \]

Then the equation for \( R \) gives

\[ \frac{\partial}{\partial R} \left( R^2 \frac{\partial f}{\partial R} \right) = (l(l+1) + \frac{2mR^2}{\hbar^2}(V(R) - E))f. \]

Let \( u = Rf \), then

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 u}{\partial R^2} + (V + \frac{\hbar^2}{2mR^2}l(l+1))u = Eu, \]

where \( V = -\frac{e^2}{4\pi\varepsilon_0 r} \). This is one-dimensional Schroedinger equation. Guess a solution of the form \( u(r) = R^{l+1}e^{-\frac{\hbar}{\hbar_0}} \). Then twice differentiating \( R^{l+1} \) cancels the centrifugal part. Differentiating \( R^{l+1} \) and \( e^{-\frac{\hbar}{\hbar_0}} \), cancels \( V \), when \( \frac{\hbar^2}{m} a_0 = \frac{e^2}{4\pi\varepsilon_0} \), i.e,

\[ a_0 = \frac{(l+1)\hbar^2}{me^2}, \quad E = \frac{\hbar^2}{2ma_0^2}. \]

However, we donot have to cancel \( V \) immediately. We can add another term

\[ u(r) = R^{l+1}e^{-\frac{\hbar}{\hbar_0}} + c_1 R^{l+2}e^{-\frac{\hbar}{\hbar_0}}. \]

Then centrifugal part of second term \( c_1 \) can cancel the part of first term obtained by differentiating \( R^{l+1} \) and \( e^{-\frac{\hbar}{\hbar_0}} \). For this \( c_1 \) has to be chosen correct. Now we cancel \( V \) by differentiating \( R^{l+2} \) and \( e^{-\frac{\hbar}{\hbar_0}} \).

Then in general

\[ u(r) = R^{l+1}e^{-\frac{\hbar}{\hbar_0}} (1 + \sum_{j=1}^{d} c_j R^j), \]

with \( n = l + d + 1 \), the principle quantum number. Then

\[ \frac{\hbar^2}{m} a_0 = \frac{e^2}{4\pi\varepsilon_0}, \quad a_0 \propto n \]

and

\[ \frac{c_j}{c_{j-1}} = \frac{2(l+j-n)a_0^{-1}}{j(2l+j+1)}. \]

This gives \( a_0 \) and finally
5.1 Quantum Mechanics

\[ E = \frac{\hbar^2}{2ma_0}, \quad E \propto \frac{1}{n^2}. \]

5.1.3 Angular Momentum

\[ L = r \times p. \]

\[ L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x. \]

Using \([p_x, x] = -i\hbar\), etc, we have

\[ L^2 = L_x^2 + L_y^2 + L_z^2 = R^2(p_x^2 + p_y^2 + p_z^2) - (xp_x + yp_y + zp_z - i\hbar)^2 + \hbar^2. \]

A quick calculation shows

\[ xp_x + yp_y + zp_z = -i\hbar R \frac{\partial}{\partial R}. \]

Now substituting for

\[ p_x^2 + p_y^2 + p_z^2 = -\hbar^2 \left( \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 \frac{\partial}{\partial R}) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left( \sin \theta \frac{\partial}{\partial \phi} \right) \right). \] (5.6)

Then from Eq. 5.5,

\[ L^2 Y(\theta, \phi) = \hbar^2 l(l + 1). \]

and

\[ L_z Y(\theta, \phi) = -i\hbar \frac{\partial}{\partial \phi} Y(\theta, \phi) = i\hbar \frac{\partial}{\partial \phi} Y(\theta, \phi). \]

We denote the eigenfunction as \( Y_{lm} \)

Observe easily verifiable commutation relations

\[ [L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y. \] (5.8)

Define \( L^- = L_x - iL_y \) and \( L^+ = L_x + iL_y \). \( L^- \) is called lowering operator and \( L^+ \) is called raising operator.

\[ [L^2, L^\pm] = 0, \quad [L_z, L^\pm] = \pm \hbar L^\pm. \] (5.9)

Then note \([L^2, L^\pm]Y_{lm} = 0\) implies \( L^2 L^\pm Y_{lm} = \hbar^2 l(l + 1)L^\pm Y_{lm} \) hence \( L^\pm Y_{lm} \) is a linear combination of \( Y_{lm} \) for different \( m \). Now \([L_z, L^+]Y_{lm} = L^+ Y_{lm} \) implying
\[ L_z L^+ Y_{lm} = \hbar (m + 1) L^+ Y_{lm} \] implying \[ L^+ Y_{lm} = a_m Y_{l,m+1} \]. Similarly \[ L^- Y_{lm} = b_m Y_{l,m-1} \].

Then observe \( L^+ Y_{ll} = 0 \) and \( L^- Y_{l,l} = 0 \). Furthermore

\[ [L^+, L^-] = 2\hbar L_z. \]  

Furthermore we get

\[ L^+ L^- + L^- L^+ = 2(L^2 - L_z^2). \]  

Then we get

\[ L^+ L^- = L^2 - L_z^2 + \hbar L_z \]  
\[ L^- L^+ = L^2 - L_z^2 - \hbar L_z. \]

For normalized \( Y_{lm} \) we get

\[ b_{lm} = \hbar \sqrt{l(l+1) - m(m-1)} \]  
\[ a_{lm} = \hbar \sqrt{l(l+1) - m(m+1)} \]

We talked about orbitals with principle quantum number \( n \) and integer angular momentum number \( l \) and \( z \) angular momentum \( l_z \), with \( |m| \leq l \leq n - 1 \). Here \( l \) was integer. In principle it can be half integer and is ascribed to an intrinsic angular momentum called spin. We use the quantum number \( s \) instead of \( l \). In particular \( s = \frac{1}{2} \) is called spin \( \frac{1}{2} \) a property of electron. We then have two values of \( s_z = \pm \frac{1}{2} \). Then an electron as two set of quantum numbers \( l, m \) and \( s, s_z \).

### 5.2 Fine Structure and Spin orbital coupling

We talked about spin. Let’s try to understand the physics of it. You are familiar with earth spinning on its axis. This gives earth a angular momentum. Now imagine our earth was charged. Then spinning will give earth a magnetic moment. Imagine a loop of wire carrying current (circulating charge), then it has a magnetic moment \( M = I A \), where \( I \) is the current and \( A \) area of the loop, from your basic physics. Now imagine a charge \( q \) going around in a loop of radius \( r \), with angular velocity \( \omega \). Then it makes \( \frac{\omega}{2\pi} \) rotations per sec. The current is then \( \frac{q \omega}{2\pi} \) and its magnetic moment is \( \mu_S = \frac{q \omega r^2}{2\pi} = \frac{q \omega m}{2\pi} (mvr) \) where \( l = mvr \) is the angular momentum. Then \( \mu_S = \frac{q \omega m}{2\pi} L \), the ratio \( \gamma = \frac{q \omega}{m} \) is called the gyromagnetic ratio, it relates angular momentum to magnetic moment. For reasons coming from relativity we in fact have \( \gamma = \frac{q}{m} \).

There is coupling between electron spin and orbital angular momentum. There is coupling Hamiltonian of the form
5.2 Fine Structure and Spin orbital coupling

Fig. shows an atomic orbital and an electron with an inner orbital that constitutes its spin angular momentum.

\[ H_{so} = \beta \mathbf{L} \cdot \mathbf{S}. \]  

(5.16)

Let us see how this coupling arises. When electron is at a certain point on its orbital it has a velocity \( v \) and momentum \( p \). From perspective of the electron the nucleus is moving in the opposite direction with same magnitude of velocity. Then from Biot Savart law the moving nucleus produces a magnetic field on the site of electron given by

\[ B = \frac{e \mu_0}{4\pi} \frac{p \times r}{mr^3} = \frac{e \mu_0}{4\pi} \frac{L}{mr^3}. \]  

(5.17)

The energy of the electron in this field is

\[ B \cdot \mu_s = \gamma B \cdot S = \frac{e^2 \mu_0}{4\pi m^2 r^3} L \cdot S = \frac{e^2}{4\pi \epsilon_0 c^2 m^2 r^3} L \cdot S. \]  

(5.18)

Thus \( \beta = \frac{e^2}{8\pi \epsilon_0 c^2 m^2 r^3} \). Due to phenomenon called Thomas precession, \( \beta \) is called by another factor of 2 and \( \beta = \frac{e^2}{8\pi \epsilon_0 c^2 m^2 r^3} \).

In presence of this Hamiltonian our orbitals will change. let us compute how the orbitals change and what are the new energies.

\[ L \cdot S = L_z S_z + L_x S_x + L_y S_y = L_z S_z + \frac{L^+ S^- + L^- S^+}{2}. \]  

(5.19)

For this define a new operator

\[ J^2 = (L+S)^2 = L^2 + S^2 + 2L \cdot S. \]  

(5.20)
Given $l$ and $s$, we start with the state $l_z = l$ and $s_z = s$. Denote this state by $(l, s)$. This state is an eigenstate of the operator $L \cdot S$ with eigenvalue $l, s$ and hence it is an eigenstate of $J^2$ with eigenvalue $j(j + 1)$ with $j = l + s$. Now as before we can apply lowering operator. From last section $J^-(j, j_z) = b(j, j_{z-1})$ with $b = \hbar \sqrt{j(j + 1) - j_z(j_z - 1)}$, so by applications of $J^-$ we decrease $j_z$ until it is $-j$. Hence we have constructed $2j$ or $2j + 1$ orbitals depending on if $j$ is integer or half integer.

Observe $J^-(l, s) = (l - 1, s) + (l, s - 1)$. There is another orthogonal state $e_1 = (l - 1, s) - (l, s - 1)$ which is eigenfunction of $J_z$ with eigenvalue $l + s - 1$ and hence must be an eigenfunction of $J^2$. We eigenvalue of $J^2$ cannot be $j(j + 1)$ as we have exhausted all these vectors as $J^+ e_1 = 0$. Only possible value of $J^2$ is $(j - 1)j$, we gain apply lowering operators and go from $j_z = j - 1, \ldots, -(j - 1)$.

Now we consider $J^-(l, s) = (l - 2, s) + (l - 1, s - 1) + (l, s - 2)$, which has $J_z = l + s - 2$. We have constructed two eigenvectors $J^2 = j(j + 1)$ and $J^2 = (j - 1)j$. We can form a third eigenvector, we can show it has $J^2$ value $(j - 1)(j - 2)$, we can again apply lowering operators and construct eigenvectors with $J^2$. Instead of writing $J^2$ we say this $J$ which in this case has value $j - 2$.

We start with one term $(l, s)$. Then $J^-(l, s)$ has two terms, $J^{-2}(l, s)$ has three terms. This process continues till smaller of $l, s$ say $s$ becomes $-s$. Then lowering doesn’t increase number of terms. Then starting with $j = l + s$ we go until $j = l - s$. Thus all states can be indexed by $j = l + s, \ldots, l - s$ and for a given $j$ we have $j_z = j, \ldots, -j$. Thus starting with state $(l, l_z) |s, s - z)$ we have formed state

$$\langle j, j_z \rangle = \sum_{l, l_z} c_{l_z, s} |l, l_z \rangle |s, s_z \rangle,$$

(5.22)

where as just told, $j = l + s, \ldots, l - s$ and for a given $j$ we have $j_z = j, \ldots, -j$.

In the basis $|j, j_z \rangle$, we have $L \cdot S$ is diagonal with eigenvalue $\frac{j(j + 1) - l(l + 1) - s(s + 1)}{2}$. The coefficients $c_{l_z, s}$ are called Clebsch Gordon coefficients. Fig. (5.3) shows how $n = 2$, $p$ orbital gets split due to fine structure.

![Fig. 5.3](image.png)

As we can see in the figure. A energy level $n = 1, l = 1$ in presence of $L \cdot S$ coupling gets split into two set of orbitals $j = \frac{3}{2}$ with $(j_z = \frac{1}{2}, \ldots, -\frac{1}{2})$ and $j = \frac{1}{2}$ with $(j_z = \frac{1}{2}, \ldots, -\frac{1}{2})$ with different energies. This is called fine-structure. If we estimate how big this is it is $\beta = \frac{\hbar^2 e^2}{4\pi \epsilon_0 m_e r^3} \sim 10^2 eV \sim 10^3 GHz$. It arises because
the angular momentum of the orbital and the spin of the electron talk to each other. Evaluating spin orbit coupling,

$$\langle r^3 \rangle = \frac{1}{n^3 l(l + \frac{1}{2})(l + 1)a^3}$$  \hspace{1cm} (5.23)

$$E_{so} = \alpha^4 mc^2 \frac{j(j + 1) - l(l + 1) - \frac{3}{4}}{4n^3 l(l + \frac{1}{2})(l + 1)}$$  \hspace{1cm} (5.24)

Electron has a spin, so does the nucleus of the atom. It is called nuclear spin. We denote nuclear spin with $I$ like we denote electron spin with $S$. We assume that we again have an interaction between nuclear spin and electron orbital and spin angular momentum as

$$I \cdot (L + S) = I \cdot J.$$  \hspace{1cm} (5.25)

What was between $L$ and $S$ is between $I$ and $J$ so we can define the total angular momentum

$$F = I + J.$$  \hspace{1cm} (5.26)

Given $i$ and $j$ the coupling gives $f$ taking values between $i + j, \ldots, |i - j|$. Thus a $j$ orbital gets split into $f$ orbitals. This is called hyperfine splitting. The eigenvalues of $I \cdot J$ takes one values $\frac{f(f + 1) - j(j + 1) - i(i + 1)}{2}$. Thus if we estimate how much this is, it is

$$\beta = \frac{\hbar^2 e^2 \mu_0}{4\pi m_e m_p r^3} \sim 1GHz,$$

where $m_p$ is proton mass which is $10^3$ heavier than electron mass.

---

**Fig. 5.4** Fig. A shows hyperfine levels for sodium. Fig. B shows hyperfine levels for Cesium.
5.3 Relativistic Correction

In Schrödinger equation we used kinetic energy as \( \frac{p^2}{2m} \). If we use relativistic formula of

\[
E = \sqrt{(pc)^2 + (mc^2)^2} \sim mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2}.
\]

Then we find that

We have correction to energy

\[
\Delta E_{\text{rel}} = -\frac{p^4}{8m^3c^2}.
\]

Using the fact \( \frac{p^2}{2m} = E - V \), on an orbital we can calculate

\[
\langle \Delta E_{\text{rel}} \rangle = -\frac{1}{2mc^2} E_n^2 - 2E_n\langle V \rangle + \langle V^2 \rangle.
\]

\[
E_n = -\frac{\alpha^2 mc^2}{2n^2}; \quad \alpha = \frac{e^2}{4\pi \varepsilon_0 \bar{h} c}.
\]

Using \( V = \frac{e^2}{4\pi \varepsilon_0 r} \),

\[
\langle \frac{1}{r} \rangle = \frac{1}{n^2a}, \quad \langle \frac{1}{r^2} \rangle = \frac{1}{n^3(l+\frac{1}{2})a^2}
\]

where \( a \) is Bohr radius. Putting everything together we find

\[
\langle \Delta E_{\text{rel}} \rangle = -\frac{\alpha^4 mc^2}{4n^4} \left( \frac{2n}{l+\frac{1}{2}} - \frac{3}{2} \right).
\]

Adding Eq. (5.33) and (5.24) we get what is called fine structure

\[
\langle \Delta E_{\text{fs}} \rangle = -\frac{\alpha^4 mc^2}{4n^4} \left( \frac{2n}{j+\frac{1}{2}} - \frac{3}{2} \right).
\]

5.4 Lamb Shift

5.5 Positronium

Positronium (Ps) is a system consisting of an electron and its anti-particle, a positron, bound together into an exotic atom, specifically anonium. The system is unstable: the two particles annihilate each other to predominantly produce two or three gamma-rays, depending on the relative spin states. The orbit and energy levels
of the two particles are similar to that of the hydrogen atom (which is a bound state of a proton and an electron). However, because of the reduced mass, the frequencies of the spectral lines are less than half of the corresponding hydrogen lines.

**Fig. 5.5**

**Fig. 5.6 Quarkonium**

In positronium the various combinations of angular momentum cause only minuscule shifts in energy (shown by expanding the vertical scale). At 6.8 electron volts positronium dissociates.

**Fig. 5.7**

At 633 MeV above the energy of the \( J^{+} \) charmonium becomes quasi-bound, because it can decay into \( D^{0} \) and \( \bar{D}^{0} \) mesons. (From "Quarkonium."

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References