Chapter1: Linear Systems

1 Peano Baker Series

Solution of n dimensional system $\dot{x} = A(t)x$ is given by $x(t) = \Phi(t, t_0)x(t_0)$, where,

$$\phi(t, t_0) = I + \int_{t_0}^{t} A(\sigma_1)d\sigma_1 + \int_{t_0}^{t}\int_{t_0}^{\sigma_1} A(\sigma_1)A(\sigma_2)d\sigma_2 d\sigma_1 + \ldots ...$$

We show the above series converges. For $\sigma \in [t_0, t]$, let $a = max|A_{ij}(\sigma)|$. Then using $|BC| \leq nbc$, where $b = max|B_{ij}|$, we get

$$\begin{aligned}
|\Phi(t, t_0)_{ij}| &\leq 1 + \frac{1}{n}\ (\ na(t - t_0) + (na)^2(t - t_0)^2/2 + (na)^3(t - t_0)^3/3 + ...) && (1)\\
&\leq 1 + \frac{1}{n}(\ \exp(na(t - t_0)) - 1\ ) && (2)
\end{aligned}$$

$$\begin{aligned}
\Phi(t + \Delta t, t_0) - \Phi(t, t_0) &= \int_{t}^{t+\Delta t} B(\sigma)d\sigma && (3)\\
B(\sigma) &= A(\sigma) + A(\sigma)\int_{t_0}^{\sigma} A(\sigma_1)d\sigma_1 + \ldots && (4)\\
\lim_{\Delta t \to 0} \frac{\Phi(t + \Delta t, t_0) - \Phi(t, t_0)}{\Delta t} &= B(t) = A(t)\Phi(t, t_0) && (5)\\
\frac{d}{dt}\Phi(t, t_0) &= A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I && (6)
\end{aligned}$$

Series is absolutely bounded then it converges absolutely and hence converges. $\Phi(t, t_0)$ is called the state transition matrix.

Solution is unique, else there is another solution $y(t)$, with $y(t_0) = x_0$ satifying $\dot{y}(t) = A(t)y(t)$ then for $z(t) = z(t) - y(t)$ we have $\dot{z}(t) = A(t)z(t)$, with $z(0) = 0$, then

$$\frac{d\|z\|^2}{dt} = z^T(A^T + A)z \leq \lambda\|z\|^2,$$

for $\lambda > 0$. Then for $\eta > \lambda$, we have

$$\frac{d}{dt}\exp(-\eta t)\|z\|^2 \leq 0.$$

Hence $z(t) = 0$ and $y(t) = x(t)$. Therefore we have a solution that exists for all times and is unique.

Observe

$$\dot{x} = (1 + x^2), \quad x(0) = 0$$

has solution $x(t) = \tan t$ that does not exist beyond $\frac{\pi}{2}$.

Similarly

$$\dot{x} = \sqrt{x}, \quad x(0) = 0$$

has solution $x(t) = (t - c)^2/4$ for $t \geq c$ and $x(t) = 0$ for $t < c$. Family of solutions for $c > 0$, not unique. For linear system we have existence and uniqueness.

By uniqueness, we can say that

$$\Phi(t, t_0) = \Phi(t, \tau)\Phi(\tau, t_0).$$

In particular

$$I = \Phi(t_0, t_0) = \Phi(t_0, \tau)\Phi(\tau, t_0).$$

$\Phi(t, t_0)$ is invertible with inverse $\Phi(t_0, t)$.

When A is constant

$$\Phi(t, t_0) = I + A(t - t_0) + A^2(t - t_0)^2/2 + \cdots + A^k \frac{(t - t_0)^k}{k!} + \cdots = \exp(A(t - t_0)).$$

when $A(t) = a(t)$ a scalar, then we have

$$\int_{t_0}^{t} \int_{t_0}^{\sigma_1} .. \int_{t_0}^{\sigma_k} a(\sigma_1) \dots a(\sigma_k) d\sigma_k..d\sigma_1 = \frac{(\int_{t_0}^{t} a(\sigma)d\sigma)^k}{k!}.$$

$$\Phi(t, t_0) = \exp(\int_{t_0}^{t} a(\sigma)d\sigma).$$

As some examples

1.
$$A = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}; \quad \exp(At) = \begin{bmatrix} \cos(at) & -\sin(at) \\ \sin(at) & \cos(at) \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}; \;\; \exp(At) = \begin{bmatrix} \cosh(at) & \sinh(at) \\ \sinh(at) & \cosh(at) \end{bmatrix}$$

3.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \;\; \exp(At) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

4.

$$A = \begin{bmatrix} 0 & -a(t) \\ a(t) & 0 \end{bmatrix}; \;\; \Phi(t,0) = \begin{bmatrix} \cos b(t) & -\sin b(t) \\ \sin b(t) & \cos b(t) \end{bmatrix}; \;\; b(t) = \int_0^t a(\sigma) d\sigma$$

When A and B commute $\exp((A+B)t) = \exp(At)\exp(Bt)$. They both satify the $\dot{X} = (A+B)X, \; X(0) = I$. Therefore,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \;\; \exp(At) = e^t \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

If $A(t), B(t)$ commute for all t then $\Phi_{A+B}(t,t_0) = \Phi_A(t,t_0)\Phi_B(t,t_0)$.

Given $A(t)$ and its transition matrix $\Phi(t,t_0)$, we have

$$\frac{d}{dt} det\Phi(t,t_0) = tr\,(a(t)) \;\; det\Phi(t,t_0)$$

$$\begin{aligned}
\frac{d}{dt} det\Phi(t,t_0) &= \sum \frac{\partial \Phi(t,t_0)}{\partial \Phi_{ij}} \dot{\Phi}_{ij} \\
det\Phi(t,t_0) &= \sum c_{ij}\Phi_{ij} \\
\frac{d}{dt} det\Phi(t,t_0) &= \sum c_{ij}\dot{\Phi}_{ij} = tr(A\Phi C^T) = tr(A) det\Phi \\
det\Phi(t,t_0) &= \exp(\int_{t_0}^t tr(A(\sigma))d\sigma).
\end{aligned}$$

2 Variation of Constant Formula

Now consider the linear control system,

$$\dot{x} = A(t)x + \underbrace{B(t)}_{n\times m}\underbrace{u(t)}_{m\times 1}; \;\; x(0) = x_0 \tag{7}$$

The solution is

$$x(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)B(\tau)u(\tau)d\tau. \tag{8}$$

Observe

$$\frac{d}{dt}\int_0^t \Phi(t,\tau)B(\tau)u(\tau)d\tau = A(t)\int_0^t \Phi(t,\tau)B(\tau)u(\tau)d\tau + B(t)u(t).$$

When A and B are constant variation of constant formula is

$$x(t) = \exp(At)x_0 + \int_0^t \exp(A(t-\tau))Bu(\tau)d\tau. \tag{9}$$

Consider a force harmonic oscillator

$$\frac{d^2}{dt^2}x(t) + x = u(t),$$

driven by a sinusoidal input $u(t) = \sin\omega t$, with $x(0) = \dot{x}(0) = 0$.

Writing $x_1 = x$ and $x_2 = \dot{x}$, we have,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B}u(t).$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}}_{\exp(At)}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{bmatrix}\begin{bmatrix} 0 \\ \sin\omega t \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2}\int_0^t \begin{bmatrix} \cos(-\omega\tau + t - \tau) - \cos(\omega\tau + t - \tau) \\ \sin(\omega\tau + t - \tau) - \sin(-\omega\tau + t - \tau) \end{bmatrix} d\tau.$$

when $\omega = 1$,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sin t - t\cos t \\ t\sin t \end{bmatrix}.$$

The solution grows with t. This is excitation on resonance. Linear systems are ubiquitous. Electrical, mechanical systems.

3 Controllability

Consider

$$\dot{x} = B(t)u(t), \quad x(t) = x_0 + \int_0^t B(\tau)u(\tau)d\tau.$$

Where all can you drive the system starting from x_0. Let

$$W = \int_0^t B(\tau)B^T(\tau)d\tau.$$

$x(t)$ is reachable iff $y = x(t) - x(0)$ is in the range space of W denoted as $\mathcal{R}(W)$.

Proof: If $y \in \mathcal{R}(W)$ then $y = W\xi = \int_0^t B(\tau)B^T(\tau)d\tau\xi$. Let $u(t) = B^T(\tau)\xi$.
If $y \notin \mathcal{R}(W)$, then $y = z + a$ where $z \in \mathcal{R}(W)^\perp$ and $a \in \mathcal{R}$ and $z \neq 0$. Then $\exists z \neq 0$ such that $\langle z, y\rangle \neq 0$ and $z \in \mathcal{R}(W)^\perp$. Then $\int_0^t z^T B(\tau)u(\tau)d\tau \neq 0$. But $\int_0^t z^T B(\tau)B^T(\tau)zd\tau = 0 \to z^T B(\tau) = 0$. Hence contradiction. Therefore $y \in \mathcal{R}(W)$.

The theorem has at it essence that given matrix P, $\mathcal{R}(P)$ is same as $\mathcal{R}(PP^T)$. Given $y = \int_0^T B(\tau)u(\tau)d\tau$, we can write this as $y = PU$, where define discrete time approximation with $\Delta t = T/n$, we have

$$P = \sqrt{\Delta t}\left[\begin{array}{c|c|c|c} B(n\Delta t) & B((n-1)\Delta t) & \dots & B(\Delta t)\end{array}\right],\; U = \sqrt{\Delta t}\left[\begin{array}{c} u(n\Delta t) \\ \hline u((n-1)\Delta t) \\ \hline \dots \\ \hline u(\Delta t)\end{array}\right].$$

Now we come to the general linear system.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), x(0) = x_0$$

and ask when is x_f reachable at time T. Then it is reachable if we have $u(\tau)$ such that

$$x_f - \Phi(T,0)x_0 = \int_0^T \Phi(T,\tau)B(\tau)u(\tau)d\tau.$$

Then from the above results $x_f - \Phi(T,0)x_0$ is reachable at time T is it lies in rangle space of W, where

$$W = \int_0^T \Phi(T,\tau)B(\tau)B^T(\tau)\Phi(T,\tau)^T d\tau.$$

W is called controllability grammian. Everypoint can be reached at time T, if controllability Grammian is full rank. When A and B are constant then

$$W = \int_0^T \exp(A(T-\tau))BB^T \exp(A(T-\tau))^T d\tau = \int_0^T \exp(At)BB^T \exp(At)^T dt.$$

Let

$$M = \left[\ B \,\middle|\ \ AB \,\middle|\ \ \ldots \,\middle|\ \ A^{n-1}B\ \right].$$

We claim

$$\mathcal{R}(W) = \mathcal{R}(MM^T).$$

Recall by Cayley Hamilton theorem A satifies its n^{th} order characteristic polynomial. Which helps to express A^n in lower powers of A. Then we have

$$\exp(At) = \sum_{i=0}^{n-1} \alpha_i(t) A^i,$$

and

$$Wx = \int_0^T \exp(At)BB^T \exp(At)^T x\ dt = \int_0^T \sum_{i=0}^{n-1} \alpha_i(t) A^i B y(t) = Mz.$$

for some z. Hence $\mathcal{R}(W) \in \mathcal{R}(M)$.

Now we show that Null space of of W denoted $\mathcal{N}(W)$ belongs to $\mathcal{N}(M^T)$. Suppose $x \in \mathcal{N}(W)$, then

$$\int_0^T \exp(At)BB^T \exp(At)^T dt\ x = 0; \qquad x^T \int_0^T \exp(At)BB^T \exp(At)^T dt\ x = 0$$

$$\int_0^T \|B^T \exp(At)^T x\|\ dt\ = 0$$

This means $\|B^T \exp(At)^T x\| = 0$ for all t. It implies $B^T x = 0$, else we make t small and $\|B^T \exp(At)^T x\| \neq 0$. Similarly we have

$$(AB)^T x = (A^2 B)^T x = \cdots = (A^{n-1}B)^T x = 0.$$

Therefore $x \in \mathcal{N}(M^T)$ and hence $x \in \mathcal{N}(MM^T)$. Because given matrix P, $\mathcal{N}(P)$ is same as $\mathcal{N}(P^T P)$. Now given a symmetrix matrix S, $\mathcal{N}(S)$ and $\mathcal{R}(S)$ are orthogonal complement of each other.

To show this we remember given P, $\mathcal{R}(P)^\perp = \mathcal{N}(P^T)$. Because if $y \in \mathcal{R}(P)^\perp$, we have $y^T P x = 0$, $\forall x$. Then $(P^T y)^T x = 0$, $\forall x$, hence $P^T y = 0$ and $y \in \mathcal{N}(P^T)$. Converse follows on same lines.

Thus we have

$$\mathcal{R}(W) = \mathcal{R}(MM^T); \;\; \mathcal{N}(W) = \mathcal{N}(MM^T).$$

Now if $M = \left[\; B \middle| \; AB \middle| \; \ldots \middle| \; A^{n-1}B \;\right]$ is full rank, the system is controllable.

Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ddots & 1 & \ldots & 0 \\ 0 & 0 & \ldots & \ddots & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & \ldots & -\alpha_k & \ldots & -\alpha_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

Then see M is full ranked. The above form is called controllable canonical form.

In general, given a controllable control system

$$\dot{x} = \underbrace{A}_{n\times n} x + \underbrace{b}_{n\times 1} u.$$

Consider the full rank matrix

$$P = \left[\; A^{n-1}b \middle| \; \ldots \middle| \; Ab \middle| \; b \;\right],$$

Let $y = P^{-1}x$.Then

$$\dot{y} = P^{-1}APx + P^{-1}bu.$$

But note

$$P^{-1}b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

$$P^{-1}AP = \begin{bmatrix} -\alpha_n & 1 & 0 & \dots & \dots & 0 \\ -\alpha_{n-1} & 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & 1 & \dots & 0 \\ -\alpha_k & 0 & \dots & \ddots & \dots & 0 \\ \vdots & \dots & \dots & \dots & 0 & 1 \\ -\alpha_1 & 0 & \dots & 0 & \dots & 0 \end{bmatrix}, \quad A^n + \sum_{i=1}^{n} \alpha_i A^{i-1} = 0$$

This is a canonical form.

Now let q be the first row of inverse of P. Then form the matrix

$$Q = \begin{bmatrix} q \\ \hline qA \\ \hline \dots \\ \hline qA^{n-1} \end{bmatrix}. \tag{10}$$

Let $z = Qx$.Then

$$\dot{z} = QAQ^{-1}x + Qbu.$$

But note

$$Qb = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

and

$$QAQ^{-1} = \begin{bmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & 1 & \dots & 0 \\ 0 & 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 \\ -\alpha_1 & -\alpha_2 & \dots & -\alpha_k & \dots & -\alpha_n \end{bmatrix}, \quad A^n + \sum_{i=1}^{n} \alpha_i A^{i-1} = 0$$

The standard controllable canonical form.

4 Least square theory

For $m \leq n$ consider $n \times m$ matrix A of rank m, and the problem of minimizing $\|Au - b\|$ for choice of u. If we decompose b as $\hat{b} + b^{\perp}$, where $\hat{b}$ is projection of b on subspace spanned by A, then

$$\|Au - b\| = \|Au - \hat{b}\| + \|b^{\perp}\|$$

Then choose u such that

$$Au = \hat{b}, \quad A^T A u = A^T \hat{b}, \quad A^T A u = A^T b \quad , u = (A^T A)^{-1} A^T b.$$

$A^T A$ invertible as A rank m. Now consider a second problem

For $m \leq n$ consider $m \times n$ matrix A of rank m, and the problem of minimizing $\|u\|$ such that $Au = b$. $u = u_0 + n$ where $Au_0 = b$ and $n \in \mathcal{N}(A)$. $\|n + u_0\| = \|n - (-u_0)\|$, then as above $n = -\hat{u_0}$ and $n + u_0 = u_0^{\perp}$, thus u lies perpendicular to null space of A and recall $\mathcal{N}^{\perp}(A) = \mathcal{R}(A^T)$. Then $u = A^T \xi$ and we get

$$AA^T \xi = b; \quad u = A^T (AA^T)^{-1} b.$$

Again AA^T invertible as A rank m.

Now consider the problem of steering the control system

$$\dot{x} = A(t)x(t) + B(t)u(t), \quad x(0) = x_0$$

to $x(T) = x_f$ and minimize $\int_0^T \|u\|^2 dt$.

Then,

$$z = x_f - \Phi(T, 0)x_0 = \int_0^T \underbrace{\Phi(T, \tau)B(\tau)}_{c(\tau)} u(\tau) d\tau.$$

As before define descrete time approximation with $\Delta t = T/n$, we have

$$C = \sqrt{\Delta t} \left[\; c(n\Delta t) \;\middle|\; c((n-1)\Delta t) \;\middle|\; \ldots \;\middle|\; c(\Delta t) \; \right], \; U = \sqrt{\Delta t} \begin{bmatrix} \dfrac{u(n\Delta t)}{\dfrac{u((n-1)\Delta t)}{\dfrac{\ldots}{u(\Delta t)}}} \end{bmatrix},$$

We want to solve minimize $U^T U$ subject to $z = CU$. This is what we just solved. The answer is $U = C^T (CC^T)^{-1} z$. which is expressed as

$$u(\tau) = B'(\tau)\Phi'(T, \tau) W^{-1} z, \quad W = \int_0^T \Phi(T, \tau)B(\tau)B'(\tau)\Phi'(T, \tau) d\tau.$$

5 Misclleneous

let X be a $n \times n$ matrix that satifies

$$\dot{X} = AX + XB^T$$

Then $X(t) = \Phi_A(t,0)X(0)\Phi_B(t,0)^T$. Gust verify by differentiation.
Note

$$\Phi_A(T,t)\Phi_A(t,0) = \Phi(T,0)$$

Differentiating both sides

$$\frac{d}{dt}\Phi_A(T,t) = -\Phi_A(T,t)A.$$

or

$$\frac{d}{dt}\Phi'_A(T,t) = -A'\Phi'_A(T,t).$$

Feedback Control Systems

6 Propotional Controller

Consider the servo control system

$$\ddot{x} + \alpha\dot{x} = u$$

we want to stabilize the system to $x = 1$. Let $y = x - 1$, then

$$\ddot{y} + \alpha\dot{y} = u, \quad \alpha > 0$$

Let $y_1 = y$ and $y_2 = \dot{y}$, then

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + u\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{11}$$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} -K & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{12}$$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K & -\alpha \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{13}$$

When $K > 0$ both eigenvalues have negative real part and we stabilize the system.

7 PI controller and Speed Control

Consider the control system

$$\dot{x} + \alpha x = u$$

we want to stabilize the system to $x = 1$. Let $y = x - 1$, then

$$\dot{y} + \alpha y + \alpha = u,$$

Let $u(t) = -Ky(t) - K_i \int_0^t y(\sigma)d\sigma$
Then

$$\ddot{y} + \alpha \dot{y} = -K\dot{y} - K_i y,$$

Let $y_1 = y$ and $y_2 = \dot{y}$, then

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_i & -(\alpha + K) \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{14}$$

When $K_i > 0$ and $K + \alpha > 0$ both eigenvalues have negative real part and we stabilize the system.

7.1 speed control

Consider the servo

$$\ddot{\theta} + \alpha\dot{\theta} = u + T,$$

where T is unknown torque and we want to control to a certain speed $\dot{\theta}$. Let $x = \dot{\theta}$ then

$$\dot{x} + \alpha x = u + T$$

we want to stabilize the system to $x = 1$. Let $y = x - 1$, then

$$\dot{y} + \alpha y + \alpha = u + T,$$

Let $u(t) = -Ky(t) - K_i \int_0^t y(\sigma)d\sigma$
Then

$$\ddot{y} + \alpha \dot{y} = -K\dot{y} - K_i y,$$

and unknown torque disappears. Now everything is same as above. Suppose torque is changing with time say in discrete steps, then when we differentiate we get impulses at discrete times. These are called disturbances. We don't care now, we have a stable system.

8 PD controller

Consider the control system

$$\ddot{x} = u$$

we want to stabilize the system to $x = 1$. Let $y = x - 1$, then

$$\ddot{y} = u.$$

Let $y_1 = y$ and $y_2 = \dot{y}$, then

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + u\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{15}$$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} -K & -K_d \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{16}$$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K & -K_d \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{17}$$

When $K > 0$ and $K_d > 0$ both eigenvalues have negative real part and we stabilize the system.

9 PID controller and Servo

Consider the servo control system

$$\ddot{x} + \alpha\dot{x} + \omega^2 x = u$$

we want to stabilize the system to $x = 1$. Let $y = x - 1$, then

$$\ddot{y} + \alpha\dot{y} + \omega^2 y + \omega = u,$$

Let $u(t) = -Ky(t) - K_d\dot{y}(t) - K_i\int_0^t y(\sigma)d\sigma$

$$\dddot{y} + \alpha\ddot{y} + \omega^2\dot{y} = -K\dot{y}(t) - K_d\ddot{y}(t) - K_i y(t),$$

Let $y_1 = y$ and $y_2 = \dot{y}$, and $y_3 = \ddot{y}$ then

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_i & -(K+\omega^2) & -(\alpha+K_d) \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{18}$$

We choose K, K_i, K_d so all eigenvalues have negative real part and we stabilize the system. That is $K_i > 0$ and $\alpha + K_d > 0$ and $K + \omega^2 > 0$.

9.1 Servo

Consider the servo control system

$$\ddot{x} + \alpha\dot{x} = u + T,$$

where T is unknown torque.

we want to stabilize the system to $x = 1$. Let $y = x - 1$, then

$$\ddot{y} + \alpha\dot{y} = u + T,$$

Let $u(t) = -Ky(t) - K_d\dot{y}(t) - K_i\int_0^t y(\sigma)d\sigma$

$$\dddot{y} + \alpha\ddot{y} + \omega^2\dot{y} = -K\dot{y}(t) - K_d\ddot{y}(t) - K_i y(t),$$

everything is same as above , unknown torque disappears.

10 Pole Placement

Now consider a controllable single input system

$$\dot{x} = Ax + bu$$

Then we can always stabilize the system and place the poles of the closed loop system as desired. We can write the system in canonocal controllable form using $y = P^{-1}x$

$$\dot{y} = \begin{bmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & 1 & \dots & 0 \\ 0 & 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 \\ -\alpha_1 & -\alpha_2 & \dots & -\alpha_k & \dots & -\alpha_n \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

Let

$$u = \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_k & \dots & \beta_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \\ \vdots \\ y_n \end{bmatrix}$$

Then closed loop evolution is

$$\dot{y} = \begin{bmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & 1 & \dots & 0 \\ 0 & 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 \\ \underbrace{\beta_1 - \alpha_1}_{-\gamma_1} & \underbrace{\beta_2 - \alpha_2}_{-\gamma_2} & \dots & \underbrace{\beta_k - \alpha_k}_{-\gamma_k} & \dots & \underbrace{\beta_n - \alpha_n}_{-\gamma_n} \end{bmatrix} y$$

The characteristic polynomial of the system takes the form

$$p(s) = s^n + \gamma_n s^{n-1} + \dots + \gamma_1$$

We can choose γ_k and hence β_k are place the poles as we like. Going back to original coordinates,

$$u = \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_k & \dots & \beta_n \end{bmatrix} P^{-1} P \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \tilde{\beta}_1 & \tilde{\beta}_2 & \dots & \tilde{\beta}_k & \dots & \tilde{\beta}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{bmatrix}$$

10.1 Observers

Our feedback in previous section depended on all the state variables. In practice all state variables may not be measured. However we can be in a situation that although we don't see all state variables our system is still observable, i.e., given the controllable system

$$\dot{x} = Ax + bu; \quad y = cx, \tag{19}$$

we have

$$P = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix},$$

is full rank.

Let $P^{-1} = \begin{bmatrix} \vdots \Big| & \vdots \Big| & \dots \Big| q \end{bmatrix}$. Let $Q = \begin{bmatrix} A^{n-1}q \,|\, & A^{n-2}q \,|\, & \dots |q \end{bmatrix}$.

Then observe The transformation $z = Q^{-1}x$ gives the system

$$\dot{z} = Q^{-1}AQz + Q^{-1}bu; \quad y = cQz, \tag{20}$$

Note

$$\tilde{c} = cQ = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}.$$

and

$$Q^{-1}AQ = Q^{-1}\left[\; A^n q \Big| \;\; A^{n-1}q \Big| \;\; \dots \Big| Aq \;\right] = \underbrace{\begin{bmatrix} -\alpha_n & 1 & 0 & \dots & \dots & 0 \\ -\alpha_{n-1} & 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & 1 & \dots & 0 \\ -\alpha_k & 0 & \dots & \ddots & \dots & 0 \\ \vdots & \dots & \dots & \dots & 0 & 1 \\ -\alpha_1 & 0 & \dots & 0 & \dots & 0 \end{bmatrix}}_{R}, \;\; A^n + \sum_{i=1}^{n} \alpha_i A^{i-1} = 0.$$

Then

$$\dot{z} = Rz + \tilde{b}u; \quad y = \tilde{c}z, \tag{21}$$

Let

$$\dot{o} = Ro + \tilde{b}u + \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_n \end{bmatrix}}_{\beta} \tilde{c}(o - z); \tag{22}$$

We are free to choose β, we call o observer. It reconstructs the state z, with inputs as what can be observed about z.

Then

$$\frac{d}{dt}(z - o) = R(z - o) + \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_n \end{bmatrix}}_{\beta} \tilde{c}(z - o); \tag{23}$$

Now choose $\underbrace{R + \beta\tilde{c}}_{\bar{R}}$ so that we can place the poles and stabilize, then o aproaches z.

Now note z equation is a controllable equation so there exists a control $u = dz$ such that

$$\dot{z} = Rz + \tilde{b}dz = \tilde{R}z \tag{24}$$

is stable. We instead use the feedback

$$\dot{z} = Rz + \tilde{b}do = Rz + \tilde{b}dz + \tilde{b}d(o - z) = \tilde{R}z + \tilde{b}d(o - z) \tag{25}$$

Now writing a equation

$$\frac{d}{dt}\begin{bmatrix} z \\ z - o \end{bmatrix} = \begin{bmatrix} \tilde{R} & \tilde{b}d \\ 0 & \bar{R} \end{bmatrix}\begin{bmatrix} z \\ z - o \end{bmatrix} \tag{26}$$

This is a stable system z goes to 0.

Let $q = Qo$ and $\tilde{\beta} = Q\beta$, and $\tilde{d} = dQ^{-1}$ then note from Eq. 22 we have

$$\dot{q} = Aq + bu + \tilde{\beta}c(q - x); \tag{27}$$

and

$$\dot{x} = Ax + b\tilde{d}q; \tag{28}$$

As we have shown z goes to zero and x goes to zero.

11 Excercises

1. Let A be a $n \times n$ matrix,

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & 1 & \dots & 0 \\ 0 & 0 & \dots & \ddots & \ddots & 0 \\ \vdots & \dots & \dots & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & \dots & 0 \end{bmatrix}.$$

   Find $\exp(At)$.

2. Let A be a $n \times n$ matrix,

$$A = \begin{bmatrix} 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & 1 & \dots & 0 \\ 0 & 0 & \dots & \ddots & \ddots & 0 \\ \vdots & \dots & \dots & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & \dots & 1 \end{bmatrix}.$$

Find $\exp(At)$.

3. Let A be a constant matrix, find the state transtion matrix $\Phi(T,0)$ for the system $\dot{x} = f(t)A\,x$, where f is a continuous function.

4. Let $\Omega(t) = -\Omega^T(t)$. Show that $\Phi_\Omega(t,0)\Phi_\Omega^T(t,0) = I$.

5. Find
$$\frac{d}{dt}\exp(A + tB)|_{t=0}$$
.

6. Given $n \times n$ matrices A, B, define
$$A \otimes B = \begin{bmatrix} A_{11}B & \dots & \dots & A_{1n}B \\ A_{21}B & \dots & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1}B & \dots & \dots & A_{nn}B \end{bmatrix}.$$
Show
$$\exp(A \otimes I + I \otimes B) = \exp(A) \otimes \exp(B).$$

7. Let A be an n by n matrix. We say that a linear subspace of R^n is invariant under A if every vector x in that subspace has the property that Ax also belongs to that subspace. Show that
$$\dot{x} = Ax + \underbrace{b}_{n\times 1} u.$$
is controllable if and only if b does not belong to an invariant subspace of A.

8. Consider the scalar system $\dot{x} = x + u$. Given the constraint $u(t+T) = u(t)$, find the control that drives $x(0) = 1$ to $x(2T) = 0$ and minimizes $\int_0^{2T} u^2(t)dt$.

9. Show that Q in Eq. 10 is invertible.

10. Let $A(t+T) = A(t)$, express $\Phi_A(t,0)$ for $t > T$ in terms of $\Phi_A(\sigma,0)$ for $\sigma \leq T$.