Chapter2: Non-Linear Systems

1 Multivariable Calculus

Let $f : R^n \to R$, also written $f(x_1, x_2, \ldots, x_n)$. The partial derivative $\frac{\partial f}{\partial x_i}$ at $a = (a_1, a_2, \ldots, a_n)$ is

$$\frac{\partial f}{\partial x_i}|_a = \lim_{h \to o} \frac{f(a_1, \ldots, a_i + h, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h}.$$

f is differentiable at a if $\exists b \in R^n$, such that in a nghd U_a of a,

$$f(x) = f(a) + b^T(x - a) + \|x - a\| r(x, a)$$

such that $\lim_{x \to a} r(x, a) = 0$. If all $\frac{\partial f}{\partial x_i}$ are continuous is a nghd U_a, we say f is C^1. Note C^1 implies differentiable with $b_i = \frac{\partial f}{\partial x_i}|_a$.
proof:

$$f(a+h) - f(a) = \sum_i g_i, \;\; g_i = f(a_1, \ldots, a_{i-1}, a_i + h_i, \ldots, a_n + h_n) - f(a_1, \ldots, a_i, a_{i+1} + h_{i+1}, \ldots, a_n + h_n)$$

$$g_i = \frac{\partial f}{\partial x_i}(a_1, \ldots, a_{i-1}, a_i + \hat{h}_i, a_{i+1} + h_{i+1}, a_n + h_n) h_i$$

$$f(a + h) - f(a) = \sum \frac{\partial f}{\partial x_i}|_a h_i + \|h\| \underbrace{\sum_i (g_i - \frac{\partial f}{\partial x_i}|_a) \frac{h_i}{\|h\|}}_{r(h)}.$$

Since f is C^1 we have $\lim_{h \to 0} r(h) = 0$. Hence the proof. If all partial derivatives of order r are continuous is a nghd U_a, we say f is C^r. If partial derivatives of all order are continuous is a nghd U_a, we say f is C^∞.

Let $c(t) = f(a + th)$. Then $dc/dt|_{t=0} = \sum \frac{\partial f}{\partial x_i}|_a h_i$. My mean value theorem

$$c(1) - c(0) = f(a + h) - f(a) = \sum \frac{\partial f}{\partial x_i}|_{a + \hat{t}h} h_i, \quad \hat{t} \in [0, 1].$$

Now let $F: R^n \to R^n$, where, $F = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{pmatrix}$. We say F is differentiable when f_i are. F is C^r when f_i are. For C^1, F, we have,

$$F(a+h) - F(a) = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \dots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_a}_{DF(a)} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} + \|h\| r(h)$$

where $\lim_{h \to 0} r(h) = 0$. $DF(a)$ is called Jacobian of mapping F at a.

Lemma 1 Chain Rule: For $F, G : R^n \to R^n$, C^1, we have

$$D(F \circ G(a)) = DF(G(a)) DG(a).$$

proof: Lets see for scalar case when $f : R \to R$, C^1, we have

$$\frac{f(g(a+h)) - f(g(a))}{h} = f'(y) \frac{g(a+h) - g(a)}{h},$$

where $y \in [g(a), g(a+h)]$. Taking the $\lim_{h \to 0}$, we obtain $(f \circ g)'(a) = f'(g(a)) * g'(a)$. Now when $f : R^n \to R$, C^1, we have

$$\frac{f(G(a+h_k)) - f(G(a))}{h_k} = \sum_i \frac{\partial f}{\partial x_i}(y) \frac{g_i(a+h) - g_i(a)}{h_k},$$

where y on line joining $G(a), G(a+h_k)$. Taking the $\lim_{h \to 0}$, we obtain $\frac{\partial (f \circ G)}{\partial x_k}(a) = \sum_i \frac{\partial f}{\partial x_i}(G(a)) \frac{\partial g_i}{\partial x_k}(a)$. Then $D(F \circ G(a)) = DF(G(a)) DG(a)$.

Corollary 1 For $F : R^n \to R^n$, and $x(t) \in R^n$, we have

$$\frac{d}{dt} F(x(t)) = DF(x(t)) \dot{x}$$

1.1 Inverse Mapping Theorem

Theorem 1 For F, C^1, let $b = F(a)$. If $DF(a)$ is invertible then F maps a nghd U_a one to one and onto nghd $V_b = F(U_a)$, such that F^{-1} on V_b is C^1.

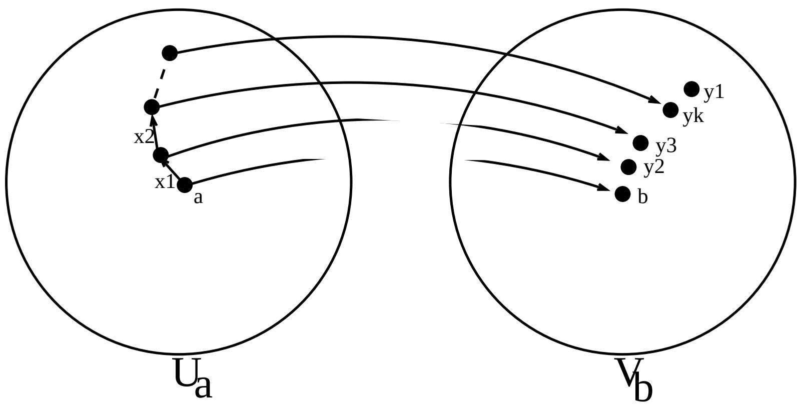


Figure 1:

Proof: Note determinant is a continuous fn. Since say $det(DF(a)) > 0$, we choose a nghd of a, $U_a(r_o)$ of radius r_o such that $det(F(x)) > \epsilon$ for $x \in U_a(r_o)$. For x and y in $U_a(r_o)$, we have $F(x) - F(y) = DF(z)(x - y)$, where z lies on line joining x and y. Since $det(DF(z)) > \epsilon$, we have F injective of $U_a(r_o)$.

Furthermore choose $U_a(r_0)$ such that $\|DF^{-1}(x)\| < \epsilon^{-1}$ and $r(x,y) < \frac{\epsilon}{2}$ for $(x,y) \in U_a(r_0)$. Note $|DF^{-1}(x)z| < \frac{\|z\|}{\epsilon}$ for all $x \in U_a(r_0)$. For $r_1 = \epsilon r_0$, we show F is onto $U_b(\frac{r_1}{2})$. For $y_1 \in U_b(\frac{r_1}{2})$, let $x_1 = DF^{-1}(a)(y_1 - b)$. Then $\|x_1\| < \frac{r_0}{2}$. Let $y_2 = F(a + x_1)$, the $\|y_1 - y_2\| < \frac{r_1}{4}$. Now define $x_2 = DF^{-1}(x_1)(y_1 - y_2)$, then $\|x_2\| < \frac{r_0}{4}$. We can continue $y_k = F(a + \sum_i^{k-1} x_i)$ and $\|y_1 - y_k\| < \frac{r_1}{2^k}$ and $x_k = DF^{-1}(x_1)(y_1 - y_k)$ and $\|x_k\| < \frac{r_0}{2^k}$. Then $F(a + \sum_i x_i)$ converges to y_1 and $a + \sum_i x_i \in U_a(r_0)$. Infact we have shown that $V_b = F(U_a(r_o))$ is open. We show $G = F^{-1}$ is continuous on V_b. At b, we can get to within r_0 of $a = G(b)$ by choosing y_1, within r_1 of b. Similarly at other points. By chain rule $DG(y) = (DF(G(y))^{-1}$. since F is C^1 and G continuous, we have G as C^1.

1.2 Implicit Function Theorem

Let A be a $m \times n$, $(m \leq n)$ matrix of rank r, then we ask what are solutions of

$$Ax = b. \tag{1}$$

By similarity transformations we can express $A = P_1 B P_2$, where,

$$B = \begin{bmatrix} I_{r\times r} & \times \\ 0 & 0 \end{bmatrix}$$

, then above equation is written as

$$B \underbrace{\left[\frac{y_1}{y_2}\right]}_{y} = \left[\frac{c_{r\times 1}}{0}\right].$$

and $y = P_2 x$. Then we are free to choose y_2 and that determines y_1 uniquely and $x = P_2^{-1} y$. Then the solution set is parameterized by a $n-r$ dimesional space y_2. Every solution x is in one to one correspondence with y_2 which are coordinates of x.

There is important nonlinear generalization of this called implicit function theorem. Let $F : R^n \to R^m$, such that F is C^1 and $F(a) = b$. If DF is of constant rank r is a nghd U_a, then we can find a nghd $V_a \subset U_a$ and a map $G : R^n \to R^n$, that maps V_a one to one onto a nghd W of origin such that solution set S of $F(x) = b$ contained in V_a is simply

$$G^{-1}(0, y_2) = S.$$

for $(\underbrace{0}_{r}, \underbrace{y_2}_{n-r}) \in W$. We have a parameterization of S.

Proof: Writing

$$DF(a) = \left[\begin{array}{c|c} \frac{\partial F_1}{\partial a_1} & \frac{\partial F_1}{\partial a_2} \\ \hline \frac{\partial F_2}{\partial a_1} & \frac{\partial F_2}{\partial a_2} \end{array}\right],$$

where $a = (\underbrace{a_1}_{r}, \underbrace{a_2}_{n-r})$, has a_1 as first r cordinates and a_2 as last $n-r$ cordinates. Similarly the map $F = (\underbrace{F_1}_{r}, \underbrace{F_2}_{n-r})$, where $F(a) = (\underbrace{b_1}_{r}, \underbrace{b_2}_{m-r})$. Since $DF(a)$ is rank r, W.L.O.G we assume that the top-left, $r \times r$ block is non-singular. We can find a open nghd around a on which top-left block is non-singular. Now consider the map $G(x_1, x_2, \ldots, x_n) = (f_1, \ldots, f_r, x_{r+1}, \ldots, x_n)$. Then

$$DG(a) = \left[\begin{array}{cc} \frac{\partial F_1}{\partial a_1} & \frac{\partial F_1}{\partial a_2} \\ 0 & I \end{array}\right],$$

which is full rank and hence by inverse mapping theorem we can find a nghd V_a such that G maps one-one onto a nghd W of $G(a)$. Now observe S in V_a maps to plane $(b_1, \cdot)$ in W, furthermore in W, $G^{-1}(b_1, \cdot) = S$. In W, given a point on the plane $(b_1, \cdot)$, look at it preimage z in V_a and join it by a curve $\mathcal{C}$ to a. On V_a, the last $n-r$ rows of $DF(x)$ are depndent on first r rows. Since the integral of first r rows along $\mathcal{C}$ is zero so is true for last $n-r$ rows. Hence $F(z) = b$ and thus $z \in S$. Thus we have a parametrization of S, the plane $(b_1, \cdot)$ in W.

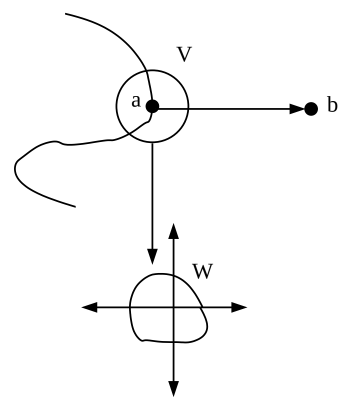


Figure 2:

2 Manifolds

In implicit function theorem, we saw how the solution set S of $F(x) = b$ has a local parametrization $G^{-1}(b_1, \cdot)$. This is called a manifold $\mathcal{M}$. When around every point, we can find a nghd U and a map ϕ that maps U one-one, onto a nghd V of orgin such that in U, $\mathcal{M} = \phi^{-1}(\underbrace{0}_{n-r}, \underbrace{\cdot}_{r})$, for $(\underbrace{0}_{n-r}, \underbrace{\cdot}_{r}) \in V$. We say we have local coordinates for $\mathcal{M}$. r is called dimension of $\mathcal{M}$.

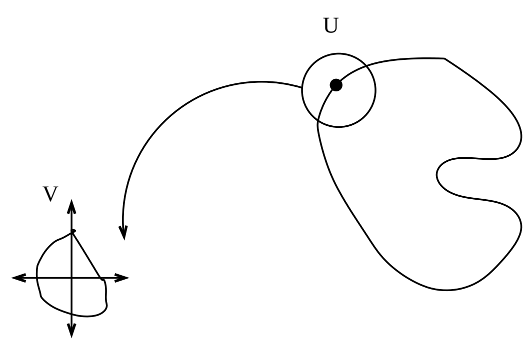


Figure 3:

2.1 Examples

Sphere: Let $X = (x, y, z)$ satify $F(x, y, z) = x^2 + y^2 + z^2 = 1$, we show $\mathcal{M}$ is a manifold. Consider $DF(X) = 2\begin{bmatrix} x & y & z \end{bmatrix}$. Then DF(X) is rank 1 in a nghd of every $X \in \mathcal{M}$ and by implicit function theorem $\mathcal{M}$ is a manifold of dimension $3 - 1 = 2$.

Orthogonal Group O(n): Let X be a $n \times n$ real matrix satisfying $F(X) = X^T X = I$. Then at nonsingular X, we can write a velocity vector $\dot{X}(t) = AX(t)$ for $A \in n \times n$ matrices. Then $\frac{dF}{dt} = X^T(A^T + A)X$. The null space is $A^T + A = 0$. all skew symmetric matrices of dim $\frac{n(n-1)}{2}$. Then rank is dim $\frac{n(n+1)}{2}$ and by implicit function theorem we have a manifold of dim $\frac{n(n-1)}{2}$. Its called the Orthogonal group.

Special Orthogonal Group SO(n): Let $X \in O(n)$ the $detX = \pm 1$. Then X has two disconnected component $detX = 1$ and $detX = -1$. The component with $detX = 1$ is called $SO(n)$. Its called the special Orthogonal group.

Unitary Group U(n): Let X be a $n \times n$ complex matrix satisfying $F(X) = X'X = I$. Then at nonsingular X, we can write a velocity vector $\dot{X}(t) = AX(t)$ for $A \in n \times n$ complex matrix. Then

$$\frac{dF}{dt} = X'(A' + A)X.$$

The null space is $A' + A = 0$, all skew hermitian matrices of dim n^2. Then rank is dim n^2 and by implicit function theorem we have a manifold of dim $2n^2 - n^2 = n^2$. Its called the Unitary group.

Special Unitary Group SU(n): Let X be a $n \times n$ complex matrix satisfying $F(X) = \begin{bmatrix} X'X \\ det(X) \end{bmatrix} = \begin{bmatrix} I \\ 1 \end{bmatrix}$. Then at nonsingular X, we can write a velocity vector $\dot{X}(t) = AX(t)$ for $A \in n \times n$ complex matrix. Then

$$\frac{dF}{dt} = \begin{bmatrix} X'(A' + A)X \\ tr(A)det(X) \end{bmatrix}.$$

The null space is $A' + A = 0$, all skew hermitian matrices and $tr(A) = 0$ of dim $n^2 - 1$. Then rank is dim $n^2 + 1$ and by implicit function theorem we have a manifold of dim $2n^2 - (n^2 + 1) = n^2 - 1$. Its called the special Unitary group.

Special Linear group SL(n, R): Let X be a $n \times n$ real matrix satisfying $detX = 1$. Then at nonsingular X, we can write a velocity vector $\dot{X}(t) = AX(t)$ for $A \in n \times n$ complex matrix. Then

$$\frac{dF}{dt} = tr(A)X.$$

The null space is $tr(A) = 0$, all traceless matrices of dim $n^2 - 1$. Then rank is dim 1 and by implicit function theorem we have a manifold of dim $n^2 - 1$.

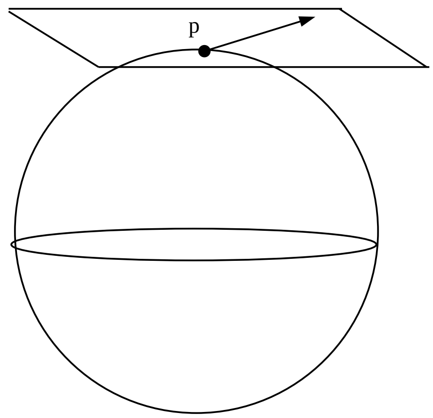
Symplectic Group Sp(n, R): Let X be a $2n \times 2n$ real matrix satisfying $X'JX = J$, where $J = \begin{bmatrix} O & -I_n \\ I_n & 0 \end{bmatrix}$. Then at nonsingular X, we can write a velocity vector $\dot{X}(t) = AX(t)$ for $A \in n \times n$ matrices. Then $\frac{dF}{dt} = X^T(A^TJ + JA)X$. The null space is $A^TJ + JA = 0$. If $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, then $A_1 = -A_4^T$ and $A_2 = A_2^T$ and $A_3 = A_3^T$. The dim of null space is $n^2 + n(n+1) = 2n^2 + n$. Then rank is dim $2n^2 - n$ and by implicit function theorem we have a manifold of dim $2n^2 + n$.

2.2 Tangent Space of $\mathcal{M}$

Given a point $p \in \mathcal{M}$, we have a nghd U_p mapped by ϕ to $\phi : U \to V_0$ such that $\mathcal{M}$ in U_a is mapped to plane $(\underbrace{0}_{n-r}, y_1, \ldots, y_r)$ in V_0, which we denotes as $(0, y)$. Consider the curve $(0, y(t))$, which is mapped to $x(t) = \phi^{-1}(0, y(t))$, passing through p. Then $\dot{x} = D\phi^{-1}(0, \dot{y})$, also denoted as $\phi_*^{-1}(0, \dot{y})$. Observe $\dot{y}$ lies in r dimensional vector space so $\dot{x} = D\phi^{-1}(0, \dot{y})$ lies in a r dimensional subspace called the tangent space of p denotes by $T_p\mathcal{M}$. Lets compute the tangent space of various manifolds.

Sphere: Let $\mathcal{M}$ be (x, y, z) satifying $x^2 + y^2 + z^2 = 1$. Given a p on $\mathcal{M}$ a curve through p satisfies $x^2(t) + y^2(t) + z^2(t) = 1$, then we have $x\dot{x} + y\dot{y} + z\dot{z} = 1$. Then $(\dot{x}, \dot{y}, \dot{z})$ is orthogonal to p, a two dimensional space.
Orthogonal Group O(n): $X \in \mathcal{M}$ is a $n \times n$ real matrix satisfying $X^TX = I$. Given a p on $\mathcal{M}$ a curve through p satisfies $X^T(t)X(t) = I$, with $X(0) = p$. We can write a velocity vector $\dot{X}(t) = AX(t)$ for $A \in n \times n$ matrix. Then $\dot{X}^T(t)X(t) + X^T\dot{X}(t) = 0$, i.e., $X^T(A^T + A)X = 0$ implying $A^T + A = 0$. A is skew symmetric matrix. The tangent space at $p = X(0)$ is of the form $AX(0)$ where A is skew symmetric matrix. Dim of $T_p\mathcal{M}$ is $\frac{n(n-1)}{2}$.

Unitary Group U(n): $X \in \mathcal{M}$ is a $n \times n$ complex matrix satisfying $X'X = I$. Given a p on $\mathcal{M}$ a curve through p satisfies $X'(t)X(t) = I$, with $X(0) = p$. We can write a velocity vector $\dot{X}(t) = AX(t)$ for $A \in n \times n$ matrix. Then $\dot{X}'(t)X(t) + X'\dot{X}(t) = 0$, i.e., $X'(A' + A)X = 0$



implying $A' + A = 0$. A is skew hermetian matrix. The tangent space at $p = X(0)$ is of the form $AX(0)$ where A is skew hermitian matrix. Dim of $T_p\mathcal{M}$ is n^2.

Special Unitary Group SU(n):$X \in \mathcal{M}$ is a $n \times n$ complex matrix satisfying $X'X = I$ and $detX = 1$. Given a p on $\mathcal{M}$ a curve through p satisfies $X'(t)X(t) = I$, $detX(t) = 1$ with $X(0) = p$. We can write a velocity vector $\dot{X}(t) = AX(t)$ for $A \in n \times n$ matrix. Then $\dot{X}'(t)X(t) + X'\dot{X}(t) = 0$, and $tr(A)detX = 0$ i.e., $X'(A' + A)X = 0$ implying $A' + A = 0$ and $tr(A) = 0$. A is traceless skew hermitian matrix. The tangent space at $p = X(0)$ is of the form $AX(0)$ where A is traceless skew hermitian matrix. Dim of $T_p\mathcal{M}$ is $n^2 - 1$.

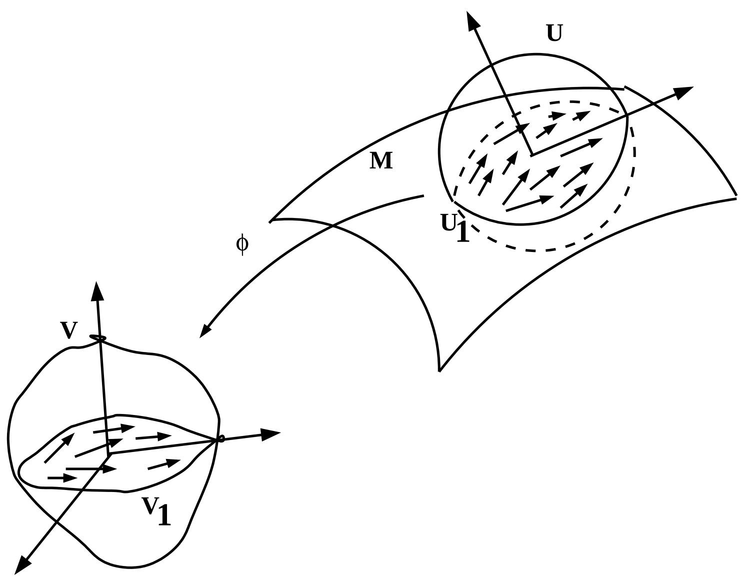
Special Linear group SL(n, R): $X \in \mathcal{M}$ is a $n \times n$ real matrix satisfying $detX = 1$. Given a p on $\mathcal{M}$ a curve through p satisfies $detX(t) = 1$ with $X(0) = p$. We can write a velocity vector $\dot{X}(t) = AX(t)$ for $A \in n \times n$ real matrix. Then $\frac{d}{dt}detX = tr(A)detX = 0$ i.e., $tr(A) = 0$. The tangent space at $p = X(0)$ is of the form $AX(0)$ where A is traceless real matrix. Dim of $T_p\mathcal{M}$ is $n^2 - 1$.

Symplectic Group Sp(n, R): $X \in \mathcal{M}$ is a $2n \times 2n$ real matrix satisfying $X'JX = J$, where $J = \begin{bmatrix} O & -I_n \\ I_n & 0 \end{bmatrix}$. Given a p on $\mathcal{M}$ a curve through p satisfies $X'(t)JX(t) = J$, with $X(0) = p$. We can write a velocity vector $\dot{X}(t) = AX(t)$ for $A \in n \times n$ matrices. Then $X^T(t)(A^TJ + JA)X(t) = 0$, i.e., $A^TJ + JA = 0$. If $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, then $A_1 = -A_4^T$ and $A_2 = A_2^T$ and $A_3 = A_3^T$. The dim of these matrices is $n^2 + n(n+1) = 2n^2 + n$.

3 Lie Groups

The manifolds like $O(n), SO(n), Sl(n,R), U(n), SU(n), Sp(n,R)$ besides being manifolds are also groups under matrix multiplication and are closed under multplication and inverse. In particlar $\mathcal{M}$ has identity element I. They are called Lie Groups and denoted by G. As an example, let $G = Sp(n,R)$. If $X, Y \in G$, then $(XY)^T JXY = J$ and $(X^{-1})^T JX^{-1} = J$.

Lie Algebra: If we look at tangent space at X, it takes the form AX, where A belongs to a vector space, lets call $\mathfrak{g}$. Then tangent space at I, is simply A. It is no coincidence that for the group G, the tangent space at Identity and at any arbitrary element X are related by left multiplication by X. This is because if $p(t)$ is curve passing through I ($p(0) = I$) then $Y(t) = p(t)X$ is curve passing through X, then if $\dot{p}(0) = A$ lies in a vector space, then $\dot{Y}(0) = AX$. Then $\mathfrak{g}$ is called the Lie algebra of G.



Vector Field: Now let $A \in \mathfrak{g}$ and consider the differential eq. $\dot{X} = AX$, with $X(0) = I$. Then $X(t) \in G$. Consider a nghd U of I in $\mathcal{M}$ and U_1 is its intersection with $\mathcal{M}$. Let $V = \phi(U)$ and $V_1 = \phi(U_1)$ is the horizontal plane, the coordinates of $\mathcal{M}$. U_1 is called nghd of $I \in \mathcal{M}$ and V_1 is called its coordinate chart. Observe, $X \in U_1$, the vector AX assigns a vector at each X and is called a *vector field*. Then $\phi_*(AX) = f(y)$ is a vector field on V_1, which is horizontal. Now consider the evolution $\dot{y} = f(y)$, as $f(y)$ is horizontal, $y(t) \in V_1$ for $t \in [-\delta, \delta]$ and it preimage satisfies $x(t) \in U_1$ and $\dot{x} = AX$. Thus we can say that for

$t \in [-\delta, \delta]$, $\exp(At) \in G$ and hence $\exp(At) \in G$ for all t.

Exponential Coordinates: At $I \in G$, we have a nghd U_1 mapped to V_1, the cordinates of U_1. There is a natural choice of cordinates in G called exponential cordinates. Let A_i be a basis of r dimesional $\mathfrak{g}$. Consider the map

$$\psi(t_1, \dots, t_r) = \exp(\sum_i t_i A_i).$$

Then $\phi \circ \psi(t_1, \dots, t_r) \to V_1$ such that $\phi \circ \psi(0) = 0$. Note

$$\frac{\partial \psi}{\partial t_i}|_{t=0} = A_i.$$

Then observe

$$(\phi \circ \psi)_*(0) = \phi_* \psi_*(0) = \phi_* \left[\begin{array}{c|c|c} A_1 & \dots & A_r \end{array} \right],$$

is full rank. Thus $\phi \circ \psi$ maps onto a nghd or origin in V_1 and hence $\psi(t_1, \dots, t_r)$ maps onto a nghd U_e of $I \in G$. We call U_e exponential nghd. For $g \in G$, $U_e\ g$ is a nghd around g.

4 Non-Linear Controllability

We now start talking about nonlinear control systems. The most common ones are of the form

$$\dot{x} = \sum_{i=1}^{m} u_i(t) g_i(x), \quad x \in R^n, \ g_i : R^n \to R^n,$$

where we assume g_i are smooth functions and are called vector fields. We ask can we steer this system between points of interest by choice of $u_i(t)$. We can write the above system as

$$\dot{x} = \sum_i u_i(t) g_i(x) = \underbrace{\left[\begin{array}{c|c|c|c} g_1(x) & g_2(x) & \dots & g_m(x) \end{array} \right]}_{G(x)} \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_m \end{bmatrix}$$

$G(x)$ is a collection of vector fields. If they span a r dimensional space at each point we call it a rank r distribution. If $r = n$, we have a controllable system. We can just follow the velocity of a path. Interesting case is when $r < n$. As an example take the following system called *nonholonomic integrator* which models kinematics of a mobile robot.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} u \\ v \\ xv - yu \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ -y \end{bmatrix}}_{f} u + \underbrace{\begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix}}_{g} v.$$

Now we are in R^3 and have two dimensional distribution spanned by f, g. At each point we can move in two directions f and g. Is the system controllable ? If we switch on $u = +1$ we move along f and if we switch on $u = -1$ we move along $-f$. Similarly if we switch on $u = +1$ we move along g and if we switch on $v = -1$ we move along $-g$. Lets go along f, then g and $-f$ and $-g$. Lets evaluate what happens then We use the notation $\phi_f^{\Delta t}(x_0)$ to denote the final point as the initial point x_0 moves along f for time Δt. Then we want to find out what is $\phi_{-g}^{\Delta t} \circ \phi_{-f}^{\Delta t} \circ \phi_g^{\Delta t} \circ \phi_f^{\Delta t}(x_0)$

Observe

$$x_1 = x(\Delta t) = x_0 + f(x_0)\Delta t + \frac{1}{2}\frac{\partial f}{\partial x} f(x_0)\Delta t^2. \tag{2}$$

$$x_2 = x(2\Delta t) = x_1 + g(x_1)\Delta t + \frac{1}{2}\frac{\partial g}{\partial x} g(x_1)\Delta t^2. \tag{3}$$

$$x_3 = x(3\Delta t) = x_2 - f(x_2)\Delta t + \frac{1}{2}\frac{\partial f}{\partial x} f(x_2)\Delta t^2. \tag{4}$$

$$x_4 = x(4\Delta t) = x_3 - g(x_3)\Delta t + \frac{1}{2}\frac{\partial g}{\partial x} g(x_3)\Delta t^2. \tag{5}$$

$$x_2 = x_0 + f(x_0)\Delta t + \frac{1}{2}\frac{\partial f}{\partial x} f(x_0)\Delta t^2 + g(x_0)\Delta t + \frac{\partial g}{\partial x} f(x_0)\Delta t^2 + \frac{1}{2}\frac{\partial g}{\partial x} g(x_0)\Delta t^2 + o(\Delta t^3). \tag{6}$$

$$x_3 = x_0 + f(x_0)\Delta t + \frac{1}{2}\frac{\partial f}{\partial x} f(x_0)\Delta t^2 + g(x_0)\Delta t + \frac{\partial g}{\partial x} f(x_0)\Delta t^2 + \frac{1}{2}\frac{\partial g}{\partial x} g(x_0)\Delta t^2 - f(x_0)\Delta t - \frac{\partial f}{\partial x}(f(x_0) + g(x_0))\Delta t^2 + \frac{1}{2}\frac{\partial f}{\partial x} f(x_0)\Delta t^2 + o(\Delta t^3). \tag{7}$$

$$x_4 = x_0 + \frac{1}{2}\frac{\partial f}{\partial x} f(x_0)\Delta t^2 + \frac{\partial g}{\partial x} f(x_0)\Delta t^2 + \frac{1}{2}\frac{\partial g}{\partial x} g(x_0)\Delta t^2 - \frac{\partial f}{\partial x}(f(x_0) + g(x_0))\Delta t^2 - \frac{\partial g}{\partial x}(g(x_0)) + \frac{1}{2}\frac{\partial f}{\partial x} f(x_0)\Delta t^2 + \frac{1}{2}\frac{\partial g}{\partial x} g(x_0)\Delta t^2 + o(\Delta t^3). \tag{8}$$

$$x_4 = x_0 + (\frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g)(x_0)\Delta t^2 + o(\Delta t^3). \tag{9}$$

$$[f,g](x) = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g.$$

When we make the maneuver we proposed. We donot return back to x_0 instead we make a leading order motion in direction given by Lie bracket of f and g denoted as $[f,g]$.

For the vector fields given

$$\frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -y \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = h.$$

We can not only go in direction f and g, we can also go in direction h by making the maneuver. Then we have three independent directions of motion, which suggest we have controllability because I can move everyway.

4.1 Frobenius Theorem

We introduce the notation $\exp(tf)x_0$, evolves x_0 under f for time t or evolves x_0 under tf for time 1. In this notaion $\exp(tf)x_0 = \phi_f^t(x_0)$.

If f and g commute, then

$$\exp(tf)\exp(sg)x_0 = \exp(sg)\exp(tf)x_0 = \exp(tf + sg)x_0. \tag{10}$$

To see this if f and g commute, then flow of f preserves g, i.e.

$$g(\phi_f^t(x_0)) = (\phi_f^t)_* g(x_0). \tag{11}$$

Note

$$\frac{dg(x(t))}{dt} = \frac{\partial g}{\partial x} f. \tag{12}$$

as f and g commute, we have

$$\frac{dg(x(t))}{dt} = \frac{\partial f}{\partial x} g(x(t)), \tag{13}$$

but this is the equation of
$(\phi_f^t)_* g(x_0)$. Now consider the curve

$$x(t) = \phi_g^{-s} \phi_f^t \phi_g^s x_0 \tag{14}$$

Then,

$$\dot{x}(t) = (\phi_g^{-s})_* f(\phi_f^t \phi_g^s x_0) = f(x(t)) \tag{15}$$

$$x(t) = \phi_f^t = \phi_g^{-s} \phi_f^t \phi_g^s x_0. \tag{16}$$

hence the proof.

Similarly, consider the curves

$$x_1(t) = \exp(tf)\exp(tg)x_0, \quad x_2(t) = \exp(t(f+g))x_0.$$

$$\dot{x}_2(t) = (f+g)(x_2(t)).$$

and

$$\dot{x}_1(t) = f(x_1(t)) + \exp(tf)_* g_1(\exp(tg)x_0)$$

. From above discussion on preserving the vector fields, we have

$$\exp(tf)_* g(\exp(tg)x_0) = g(\exp(tf)\exp(tg)x_0) = g(x_1(t)).$$

Therefore $\dot{x}_1(t) = (f+g)(x_1(t))$. By uniquesness, we have,

$$\exp(tf)\exp(tg)x_0 = \exp(t(f+g))x_0.$$

Now consider a r dimensional distribution $\Delta = \{f_1, \dots, f_r\}$ such that $r < n$ and f_i commute. Then we claim we cannot go everywhere. Our motion is restricted to a r dimensional manifold.

Let $\{f_1, \dots, f_r, e_{r+1}, \dots, e_n\}$ span R^n at x_0. Consider the map

$$\Phi(t_1, \dots, t_n) = \prod_{i=1}^{r} \exp(t_i f_i) \prod_{i=r+1}^{n} \exp(t_i e_i) x_0.$$

where the map $\exp(tf)x_0$, evolves x_0 under f for time t. Then $\frac{d}{dt}\exp(tf)x_0|_0 = f(x_0)$. Then

$$\begin{aligned} \frac{\partial \Phi}{\partial t_i}(0) &= f_i(x_0), i = 1, \dots, r \\ \frac{\partial \Phi}{\partial e_i}(0) &= e_i(x_0), i = r+1, \dots, n \end{aligned}$$

By inverse mapping theorem, we have a one-one onto map that maps nghd V to nghd U of x_0 such that the plane $(t, \dots, t_r)$ to $\mathcal{M}$.

Infact,

$$\frac{\partial \Phi}{\partial t_i}(t_1, \ldots, t_n) \quad = \quad f_i(x), i = 1, \ldots, r \tag{17}$$

$$\tag{18}$$

Thus in U, $\Phi_*^{-1}(f_i)$ are horizontal vector fields in V. Hence if we consider the equation $\dot{x} = \sum_i u_i f_i$, we are always evolving horizontally in V. Then starting from 0 in V, we stay on the horizontal plane and hence $x(t)$ evoles on $\mathcal{M}$, which is called the integral manifold of vector fields $\{f_i\}$. Hence our motion is restricted to a r dimensional manifold $\mathcal{M}$.

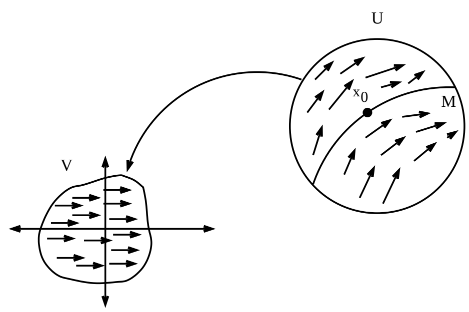


Figure 4:

Now lets relax the constraint that $\{f_i\}$ commute but instead $[f_i, f_j] \in \Delta$, i.e., at every x, $[f_i, f_j](x) = \sum_k \alpha_k(x) f_k$. Then we say our distribution Δ is *involutive*. Given a involutive distribusion Δ, there is a r dimensional manifold $\mathcal{M}$ such that Δ is tangent to $\mathcal{M}$ at each $x \in \mathcal{M}$. $\mathcal{M}$ is called the integral manifold of Δ.

Written as a $n \times r$ matrix

$$\left[\ f_1 | \ \ldots | \ \ f_r\ \right] = \begin{bmatrix} \overbrace{A}^{r\times r} \\ B \end{bmatrix}$$

These r columns are independent. At x_0, we assume. w.l.o.g the top r rows are independent. Then the top $r \times r$ matrix is invertible at x_0 and so in a nghd U of x_0. Then define

$$G = \left[\ g_1 | \ \ldots | \ \ g_r\ \right] = F * A^{-1} = \begin{bmatrix} I \\ C \end{bmatrix}. \tag{19}$$

Now observe given a function $h(x)$, we have

$$[hf, g] = h(\frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g) - (\frac{\partial h}{\partial x} g) f = h[g, f] - h_1 f$$

Therefore if $\Delta = \{f_i\}$ is involutive, we have for arbitrary functions h_i, $[h_i f_i, h_j f_j] \in \Delta$. Therefore, Eq. (19), $\{g_i\}$ is involutive and is same as Δ. Now take bracket of $[g_i, g_j]$, it is of the form $[g_i, g_j] = \begin{bmatrix} 0 \\ c(x) \end{bmatrix}$. But being involutive, it should in span of g_i, implying $c(x) = 0$ and g_i commute.

Therefore, we are back to the situation of commuting vector fields.

$$\mathcal{M} = \prod_{i=1}^{r} \exp(t_i g_i) x_0,$$

is a integral manifold passing through x_0 such that our motion under the flow $\dot{x} = \sum_i u_i f_i$ is restricted to $\mathcal{M}$. This is called *Frobenius Theorem.*

This means in our control system

$$\dot{x} = \sum_{i=1}^{r} u_i f_i(x),$$

when vector fields f_i donot span R^n and on bracketting donot generate a new vector field, then our motion is restricted to a r dimensional integral manifold $\mathcal{M}$.

4.2 Chows Theorem

Recall from Eq. (9), we calculated the map

$$\Phi_{[f,g]}^{\Delta t} x_0 = \phi_{-g}^{\Delta t} \circ \phi_{-f}^{\Delta t} \circ \phi_{g}^{\Delta t} \circ \phi_{f}^{\Delta t}(x_0).$$

We found to leading order we get

$$\Phi_{[f,g]}^{\Delta t} x_0 = x_0 + [f,g](x_0)\Delta t^2 + o(\Delta t^3).$$

To leading order we proceed in direction $[f,g]$. We say, we generate the first brackett.
Let see, how to generate second brackett, say $[h[f,g]]$. For this consider the map
$\phi_{-h}^{\Delta s}\ \Phi_{[f,g]}^{\Delta t}\ \phi_{h}^{\Delta s} x_0$,
Let $\phi_{h}^{\Delta s} x_0 = x_1$. Then we have

$$x_2 = \Phi_{[f,g]}^{\Delta t} x_1 = x_1 + \underbrace{[f,g](x_1)\Delta t^2 + o(\Delta t^3)}_{\epsilon}.$$

$$\phi_{-h}^{\Delta s}(x_1 + \epsilon) = \phi_{-h}^{\Delta s}(x_1) + \epsilon - \frac{\partial h}{\partial x}\epsilon \Delta s + o(\Delta t^2 \Delta s^2).$$

using

$$x_1 = x_0 + h(x_0)\Delta s + o(\Delta s^2)$$

$$x_2 = x_0 + [f,g](x_0)\Delta t^2 + (\frac{\partial [f,g]}{\partial x} h - \frac{\partial h}{\partial x}[f,g])(x_0)\Delta t^2 \Delta s + o(\Delta t^3) + o(\Delta t^2 \Delta s^2)$$

Now evaluate

$$\begin{aligned} \Phi^{\Delta t}_{[g,f]} x_2 &= x_2 + [g,f](x_2)\Delta t^2 + o(\Delta t^3) \\ &= x_0 + [h[f,g]](x_0)\Delta t^2 \Delta s + o(\Delta t^3) + o(\Delta t^2 \Delta s^2) \end{aligned}$$

Choosing $\Delta s = \sqrt{\Delta t}$.

$$\Psi^{\Delta t}_{[h,[g,f]]} = \Phi^{\Delta t}_{[g,f]} \phi^{\Delta s}_{-h} \ \Phi^{\Delta t}_{[f,g]} \ \phi^{\Delta s}_{h} x_0 = x_0 + [h[f,g]](x_0)\Delta t^{\frac{5}{2}} + o(\Delta t^3)$$

The leading order term is $[h[f,g]](x_0)$ a second order bracket. If we want one more bracket $[e[h[f,g]]]$, then we do

$$\phi^{\Delta s}_{-e} \Psi^{\Delta t}_{[h,[g,f]]} \phi^{\Delta s}_{e} = x_0 + [h[f,g]](x_0)\Delta t^{5/2} + [e[h[f,g]]](x_0)\Delta t^{5/2}\Delta s + o(\Delta t^3) + o(\Delta t^{5/2}\Delta s^2).$$

Now choose $\Delta s = \Delta t^{\frac{1}{4}}$, then

$$\Sigma^{\Delta t}_{[e[h[f,g]]]} = \Psi^{\Delta t}_{[-h,[g,f]]} \phi^{\Delta s}_{-e} \Psi^{\Delta t}_{[h,[g,f]]} \phi^{\Delta s}_{e} = x_0 + [e[h[f,g]]](x_0)\Delta t^{11/4} + o(\Delta t^3)$$

The leading order term is $[e[h[f,g]]](x_0)$ a second order bracket. Infact if we choose $t = (\Delta t)^{\frac{4}{11}}$, we have

$$\Sigma^{t}_{[e[h[f,g]]]} x_0 = x_0 + [e[h[f,g]]](x_0)t + o(t^{1+\alpha}), \quad \alpha > 0.$$

Now lets say we start with r independent vector fields $\{f_i\}$. By taking brackets we can generate new vector fields X_k such that $\{f_i, X_k\}$, span all of R^n. As above, we have shown how to construct a map

$$\Phi^t_X x_0 = x_0 + X(x_0)t + o(t^{1+\alpha}), \quad \alpha > 0.$$

Now consider map

$$F(t_1, \dots t_r, \dots, t_n) = \prod_{i=r+1}^{n} \Phi^{t_i}_{X_i} \prod_{i=1}^{r} \exp(f_i t_i) x_0$$

Then by construction $DF(0) = \{f_i, X_k\}$, a full rank matrix. Hence by inverse function theorem

F maps a nghd $U = (t_1, \ldots, t_n)$ of origin to a nghd V of x_0. We can go anywhere in R^n, we have controllability. Suppose we want to fo from x to y, then choose a path as shown in the figure 6 below and go from x to y by overlapping ngds. Now we can go withing ngds and go from x to y.

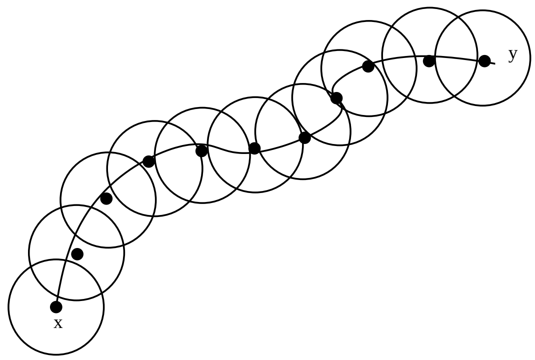


Figure 5: Figure shows how a path from x to y can be tracked through a sequence of overlapping nghds

Example 1 Let us go back to the non-holonomic integrator

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ -y \end{bmatrix}}_{f} u + \underbrace{\begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix}}_{g} v.$$

We also write

$$f = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}; \quad g = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$$

Then $h = [f, g] = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. Then f, g, h space R^3 and we have controllability.

Example 2 Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{m} \\ \dot{n} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ -y \\ 0 \\ y^2 \end{bmatrix}}_{f} u + \underbrace{\begin{bmatrix} 0 \\ 1 \\ x \\ x^2 \\ 0 \end{bmatrix}}_{g} v.$$

Then

$$[f,g] = \begin{bmatrix} 0 \\ 0 \\ 2 \\ x \\ -y \end{bmatrix}; \quad [f,[f,g]] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad [[f,g]g] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then $f, g, [f,g], [f,[f,g]], [[f,g],g]$ span the space R^5 and we have controllability.

4.3 Non-linear controllability on Lie groups

We now consider control systems of the form

$$\dot{x} = (\sum_i u_i \Omega_i)x, \quad x(0) = I$$

Where $\Omega_i \in \mathfrak{g}$ the Lie algebra of a Lie Group G. Then as shown before, $x(t) \in G$. Suppose span $\{\Omega_i\} = \mathfrak{g}$. Then sarting from say $x(0) = I$, we can go to any point in its exponential ngd U_e by just using the evolution $\exp(\sum_i u_i \Omega_i)$ for constant u_i. If G is connected any two points X and Y can be joined by a path in G. Around each point draw a exponential nghd and choose a finite cover so that we go from X to Y in overlappingnghds. Then we can go from X to Y.

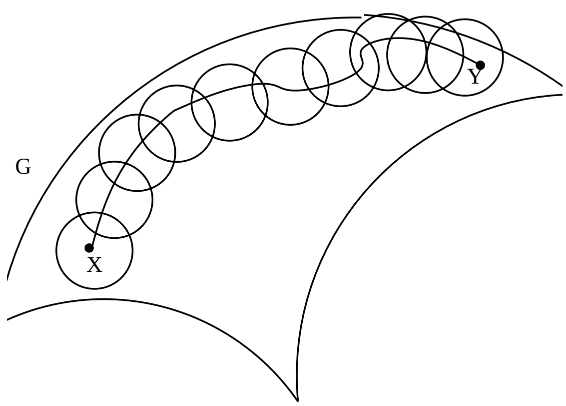


Figure 6: Figure shows how a path from x to y can be tracked through a sequence of overlapping nghds

Suppose the given Ω_i donot span $\mathfrak{g}$. Then we cannot go to all points in U_e. Again we propose the the maneuver we use before. We go in direction of Ω_1, then Ω_2 and then $-\Omega_1$ and $-\Omega_2$, for small time Δt.

This generates the evolution

$$
\begin{aligned}
U_{[\Omega_1,\Omega_2]}(\Delta t) &= \exp(-\Omega_2\Delta t)\exp(-\Omega_1\Delta t)\exp(\Omega_2\Delta t)\exp(\Omega_1\Delta t) \\
&= \exp(-\Omega_2\Delta t)\exp(-\Omega_1\Delta t)(I+\Omega_2\Delta t+\Omega_2^2\frac{\Delta t^2}{2}+o(\Delta t^3))\exp(\Omega_1\Delta t) \\
&= \exp(-\Omega_2\Delta t)(I+\Omega_2\Delta t-[\Omega_1,\Omega_2]\Delta t^2+\Omega_2^2\frac{\Delta t^2}{2}+o(\Delta t^3)) \\
&= (I-\Omega_2\Delta t+\Omega_2^2\frac{\Delta t^2}{2}+o(\Delta^3))(I+\Omega_2\Delta t-[\Omega_1,\Omega_2]\Delta t^2+\Omega_2^2\frac{\Delta t^2}{2}+o(\Delta t^3)) \\
&= I-[\Omega_1,\Omega_2]\Delta t^2+o(\Delta t^3)
\end{aligned}
$$

To leading order we generate a motion in the matrix commutator $[\Omega_2,\Omega_1]$. First note that $[\Omega_2,\Omega_1]\in\mathfrak{g}$ because for $\Delta t=\sqrt{t}$, we have $U(\sqrt{t})=I-[\Omega_1,\Omega_2]t+o(t^{3/2})$, is a path in G and its derivative at $t=0$ is an element of $\mathfrak{g}$ which is $[\Omega_2,\Omega_1]\in\mathfrak{g}$.

As before by making a maneuver, we have been able to generate a bracket. We can now generate more brackets

$$
\exp(\Delta s\Omega_3)U_{[\Omega_1,\Omega_2]}(\Delta t)\exp(-\Delta s\Omega_3)=I-[\Omega_1,\Omega_2]\Delta t^2+o(\Delta t^3)-[\Omega_3,[\Omega_1,\Omega_2]\Delta s\Delta t^2+o(\Delta s^2\Delta t^2)
$$

Choose as before $\Delta s=\sqrt{\Delta t}$. Then,

$$
U_{[\Omega_2,\Omega_1]}\exp(\Delta s\Omega_3))U_{[\Omega_1,\Omega_2]}(\Delta t)\exp(-\Delta s\Omega_3)=I-[\Omega_3,[\Omega_1,\Omega_2]\Delta t^{\frac{5}{2}}+o(\Delta t^3).
$$

Thus we have map

$$
\Phi_{[\Omega_3[\Omega_2,\Omega_1]]}(t)=1-[\Omega_3,[\Omega_1,\Omega_2]t+O(t^{1+\alpha})
$$

Thus we start with set of generators $\{\Omega_1,\dots,\Omega_r\}$. They neednot span $\mathfrak{g}$. Then we have shown how to generate brackets of Ω_i. Suppose we generate new generators call X_k such that Ω_i, X_k span $\mathfrak{g}$. Then consider the map

$$
F(t_1,\dots t_r,\dots,t_n)=\prod_{i=r+1}^{n}\Phi_{X_i}^{t_i}\prod_{i=1}^{r}\exp(\Omega_i t_i).
$$

Then by construction $DF(0)=\{\Omega_i,X_k\}=\mathfrak{g}$. Hence by inverse function theorem, F maps onto an exponential nghd U_e of I. We can go anywhere in U_e using the map F and then we have controllability as described before.

In nutshell, if commutators of the generators span $\mathfrak{g}$, we have controllability.

Example 3 Let

$$\dot{\Theta} = (u\Omega_x + v\Omega_y)\Theta, \quad \Theta(0) = I,$$

where $\Omega_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $\Omega_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. Then our system evolves on $SO(3)$, which has tangent space $\mathfrak{g} = so(3)$, 3×3 skew symmetric matrices. By Lie bracket, we get $[\Omega_x, \Omega_y] = \Omega_z$. Now we have all three generators $(\Omega_x, \Omega_y, \Omega_z)$ of $so(3)$, we span $\mathfrak{g}$. We can steer the system anywhere on $SO(3)$.

Example 4 Let

$$\dot{\Theta} = \left[\begin{array}{c|c} 0 & -u^T \\ \hline u & \mathbf{O} \end{array}\right] \Theta, \quad \Theta(0) = I,$$

where $u \in R^{n-1}$ is our control vector and $\mathbf{O}$ is $n-1 \times n-1$ matrix. First note our system evolves on $SO(n)$. Let Ω_{ij} be skew symmetric with 1 in the i,j spot, with $i < j$. Then we have $\frac{n(n-1)}{2}$ such generators that span $\mathfrak{g} = so(n)$, space of skew symmetric matrices. We only have as control, generators of the form Ω_{1k}, $k > 1$. There are $n-1$ of them. But see $[\Omega_{1j}, \Omega_{1k}] = \Omega_{jk}$. Thus we get all the generators by commutators and we have controllability.

Example 5 Let

$$\dot{\Theta} = \begin{bmatrix} 0 & u_1 & 0 & 0 & 0 \\ -u_1 & 0 & u_2 & 0 & 0 \\ 0 & -u_2 & 0 & u_3 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -u_{n-1} & 0 \end{bmatrix} \Theta, \quad \Theta(0) = I,$$

Again, our system evolves on $SO(n)$. Now we have $n-1$, control generators $\{\Omega_{12}, \Omega_{23}, \ldots, \Omega_{n-1,n}\}$. Observe $[\Omega_{12}, \Omega_{23}] = \Omega_{13}$, $[\Omega_{13}, \Omega_{34}] = \Omega_{14}$, etc. We can this way generate Ω_{1k}, and from previous example all of $so(n)$. Hence controllability.

We now return to control system

$$\dot{x} = (\sum_i u_i \Omega_i)x, \quad x(0) = I \tag{20}$$

Where $\Omega_i \in \mathfrak{g}$ the Lie algebra of a Lie Group G. $x(t) \in G$. Suppose span $\{\Omega_i\} \neq \mathfrak{g}$. Furthermore the Lie algebra generated of Ω_i denotes as $\mathfrak{h} = \{\Omega_i\}_{LA}$ is a proper subalgebra of $\mathfrak{g}$. What can we say about controllability now. Lets say A_i span $\mathfrak{h}$ and let remaining B_i span all of matrices. Consider a nghd U,

$$\Phi(t_1, \dots t_r, \dots, t_n) = \prod_{i=1}^{l} \exp(A_i t_i) \prod_{i=l+1}^{n} \exp(B_i t_i).$$

Then once again $\Phi_*^{-1}((\sum_i u_i \Omega_i)x))$ on U is a horizontal vector field and hence the flow in Eq. (20), stays on $\prod_{i=1}^{l} \exp(A_i t_i)$. This a subset of $\prod_{i=1}^{r} \exp(A'_i t_i)$. where A'_i span $\mathfrak{g}$. Hence our flow is restricted.

4.4 Control Systems with Drift

We now consider control systems of the form

$$\dot{x} = (A + \sum_i u_i B_i)x, \quad x(0) = I$$

Where $A, B_i \in \mathfrak{g}$ the Lie algebra of a Lie Group G. But now we have no control on A, its called drift as its drift of its own. We now controllability of such systems. Note before, $x(t) \in G$.

First result appears when A is periodic, i.e. this is a T such that $\exp(At) = I$, the flow of A returns you back. Then we have a similar results on controllability. If $\{A, B_i\}_{LA} = \mathfrak{g}$, we have controllability on connected Lie group G. We have to show that we can generate Lie brackets.

The main observation is that $\exp(A(T - \tau)) = \exp(-A\tau)$, we can go backwards in direction of A. Furthermore

$$\exp(A(T-\Delta))\exp((A+B)\Delta) = \exp(-A\Delta)\exp((A+B)\Delta) = I + B\Delta + O(\Delta^2) \sim \exp(B\Delta).$$

Then as before we can generate brackets of $\{A, B_i\}$ and we have controllability when $\{A, B_i\}_{LA} = \mathfrak{g}$.

Suppose A is not periodic. Then we focus on Lie groups G, which are compact (bounded), like $SO(n), SU(n)$ etc. Then given $\epsilon > 0$, there exists a time T such that $|\exp(AT) - I| < \epsilon$, i.e., if we wait long enough, we almost come back to origin. Then $\exp(A(T-\tau)) \sim \exp(-A\tau)$ and we can go backwards in direction of A. This result is based on Kronecker's theorem, which states that

Theorem 2 Kronecker: Given real number α_i, independent, i.e, $\sum_i n_i \alpha_i \neq \mathbb{Z}$ for integers n_i, not all zero. For any $\epsilon > 0$, there exist integers m_i and N such that $|\alpha_i N - m_i| < \epsilon$.

Example 6 Let

$$\dot{\Theta} = (\Omega_z + u\Omega_x)\Theta, \quad \Theta(0) = I,$$

Then our system evolves on $SO(3)$, which has tangent space $\mathfrak{g} = so(3)$, 3×3 skew symmetric matrices. Ω_z is drift which is periodic. By Lie bracket, we get $[\Omega_z, \Omega_x] = \Omega_y$. Now we have all three generators $(\Omega_x, \Omega_y, \Omega_z)$ of $so(3)$, we span $\mathfrak{g}$. We can steer the system anywhere on $SO(3)$.

Example 7 Let

$$\dot{\Theta} = (\underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{bmatrix}}_{A} + u \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{B})\Theta, \quad \Theta(0) = I,$$

Then our system evolves on $SO(3)$, which has tangent space $\mathfrak{g} = so(3)$, 3×3 skew symmetric matrices. Note drift is not periodic, but we are on a compact manifold. By Lie bracket, we get $[A, B] = \Omega_y$. Now we have all three independent generators (A, B, Ω_z) of $so(3)$, we span $\mathfrak{g}$. We can steer the system anywhere on $SO(3)$.

5 Excersises

1. **Stiefel Manifolds:** Consider for $m \leq n$, $n \times m$ real matrices Θ, such that $\Theta^T\Theta = I_m$. Show Θ is a manifold. Find it dimension and tangent space.

2. Let V be a vector space of real lower traingular matrices. Show V is an Lie algebra. Show same when V is strictly lower triangular.

3. Let us look at variant of non-holonomic integrator

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ x \end{bmatrix}}_{f} u + \underbrace{\begin{bmatrix} 0 \\ 1 \\ y \end{bmatrix}}_{g} v.$$

   Is the system controllable. Find an integral manifold passing through origin on which system lives.

4. For $i = 1, \ldots, n$, let

$$\begin{aligned} \dot{x}_i &= u_i \\ \dot{z}_{ij} &= x_i u_j - u_i x_j, i < j \end{aligned}$$

   We have n variables x_i and $\frac{n(n-1)}{2}$ valiables z_{ij}. We have n controls and $\frac{n(n+1)}{2}$ state varibles. Is the system controllable ?

5. Consider the following system with two controls

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ x_2 u_1 \\ \vdots \\ x_{n-1} u_1 \end{bmatrix}.$$

Is it controllable.

6. Consider the following system with two controls

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ x_1 u_2 \\ x_3 u_1 \\ x_3 u_2 \end{bmatrix}.$$

Is it controllable.

7. Consider the matrix equation

$$\dot{U} = -i(u\sigma_x + v\sigma_y)U, \quad U(0) = I,$$

where $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\sigma_x = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. Show that system evolves on $SU(2)$. Is it controllable.

8. Let $A = -i\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & \dots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}$. Consider the control system

$$\dot{U} = (A + \sum_{k=2}^{n} u_k \Omega_{1k})U, \quad U(0) = I$$

with skew symmetric Ω_{1k} as defined in the main text. Show system always evolves on $SU(n)$. Show it is controllable if $A \neq 0$.

9. Let $A = -i \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & \dots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \ddots \\ 0 & 0 & \dots & -1 & 0 \end{bmatrix}$ Consider the control system

$$\dot{U} = (A + uB)U, \quad U(0) = I$$

with $\lambda_{k+1} - \lambda_k \neq \lambda_{j+1} - \lambda_j$. Show system always evolves on $SU(n)$. Show it is controllable.

10. Let H and U be real symmteric matrices and Y skew symmteric. Consider the system

$$\begin{aligned} \dot{H} &= U \\ \dot{Y} = &= [H, U] \end{aligned}$$

Is the system controllable over H, Y for choice of control U.