Chapter3: Optimal Control

Until now, our focus has been on controllability. Showing we can steer our control system between points of interest. In this chapter we turn to another important question in control. How to optimally steer a dynamical system. In this chapter we will learn about Pontryagin's maximum principle.

Consider the control system

$$\dot{x} = f(x,u), \quad x \in R^n, u \in \Omega \subset R^m$$

We want to steer the system from x_0 to x_f and want to minimize

$$\eta = \int L(x,u)dt,$$

We can write this as

$$\begin{aligned} \dot{x} &= f(x,u) && (1) \\ \dot{x}_{n+1} &= L(x,u) && (2) \end{aligned}$$

and say we want to minimize x_{n+1} at final time which say is T with a control u which is optimal. We can make x, $n+1$ dimensional, (x, x_{n+1}) and write the above system as

$$\dot{x} = f(x,u) \quad (3)$$

with new f.

We do a *needle peturbations* on controls. For infinitesimal Δt, between time $(\tau - \Delta t)$ and τ, we change the control from u to v. How does the final point change. Then observe

$$\begin{aligned} \delta x(\tau) &= (f(x,v) - f(x,u))\Delta t && (4) \\ \delta x(T) &= \Phi(T,\tau)\delta x(\tau) && (5) \\ \dot{\Phi}(t,\tau) &= \underbrace{\frac{\partial f}{\partial x}|_{(x^*(t),u(t))}}_{A(t)} \Phi(t,\tau) && (6) \end{aligned}$$

Observe $\delta x(T)$ lies in a cone. If I perturb at τ_1 and τ_2, I get end point perturbation $\delta_1 x(T)$ and $\delta_2 x(T)$. Then by perturbing simulataneously for time $\alpha \Delta t$ and $(1-\alpha)\Delta t$, we get

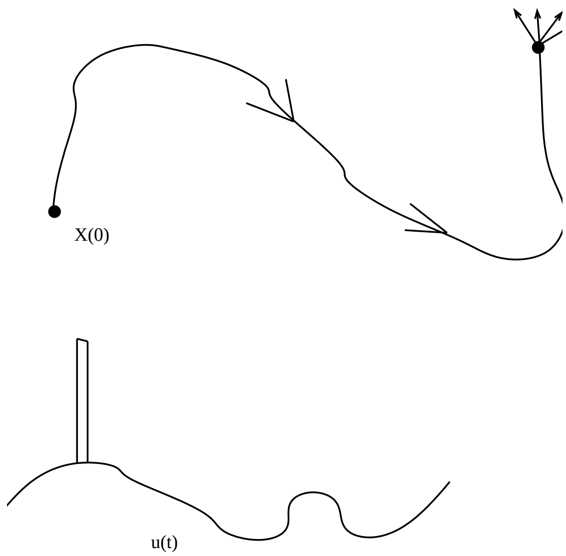

Figure 1: Figure a shows the trajectory from x_0 to x_f and how end point is perturbed as we make a needle perturbation to control in Fig. b.

$\delta x(T) = \alpha\delta_1 x(T) + (1-\alpha)\delta_2 x(T)$. Hence any convex combination is achievable. End point perturbations lie in a convex cone $\mathcal{C}$. Now we donot want that the point $p = (\underbrace{0,\ldots,0}_{n}, -1) \notin \mathcal{C}$, else we make a perturbation that fixes $x(T)$ and decreases x_{n+1}, then how is out trajectory optimal. Then we can find a hyperplane which separates p from $\bar{\mathcal{C}}$, (first assume $p \notin \bar{\mathcal{C}}$. Then there are multipliers $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$, such that $\lambda^T p < 0$ and $\lambda^T \bar{\mathcal{C}} \geq 0$. It means $\lambda_{n+1} > 0$, which can be chosen as 1 and $\lambda^T \delta x(T) \geq 0$

$$\lambda' \Phi(T,\tau)\delta x(\tau) \geq 0; \quad (\Phi(T,\tau)'\lambda)'\delta x(\tau) \geq 0$$

call $\Phi(T,\tau)'\lambda = \lambda(\tau)$, then observe

$$\dot{\lambda}(\sigma) = -A^T(\sigma)\lambda.$$

$$\lambda'(\tau)(f(x(\tau), v) - f(x(\tau), u)) \geq 0$$

Note by definition of $A(t)$ in 4, the last row of A^T is zero. Hence $\lambda_{n+1} = 1$ throughout. Then define

$$H(x(t), \lambda(t), u) = \lambda(t)' f(x, u),$$

and

$$
\begin{aligned}
H(x(t), \lambda(t), u) &\leq H(x(t), \lambda(t), v), \quad \forall v && (7)\\
\dot{\lambda}' &= -\lambda' \frac{\partial H}{\partial x} && (8)\\
\dot{x} &= (\frac{\partial H}{\partial \lambda})' && (9)
\end{aligned}
$$

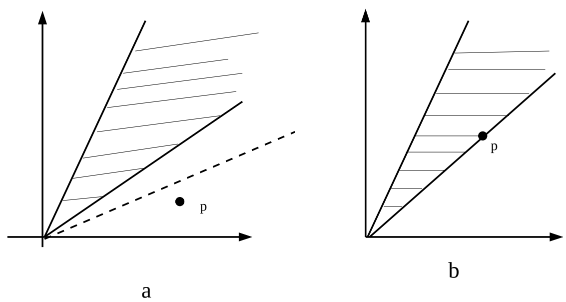

Figure 2: Figure a shows the end point cone and point p outside it which can be separated by a hyperplane. Figure b shows when p lie on boundary of $\bar{\mathcal{C}}$.

Since $\lambda_{n+1} = 1$, we can write

$$H(x(t), \lambda(t), u) = \lambda(t)' f(x(t), u) + L(x, u),$$

where λ, f refers to first n coordinates.

Observe, we assumed $p \notin \bar{\mathcal{C}}$. Such problems are called *normal problems*. It is possible that $p \in \partial \bar{\mathcal{C}}$, just on the boundary. Then recall $\lambda^T p = 0$, separating plane passes through p. Then since $p = (0, \ldots, 0, -1)$, we have $x_{n+1} = 0$ and then

$$H(x(t), \lambda(t), u) = \lambda(t)' f(x(t), u).$$

Such problems are called *abnormal*. When we are interested in minimizing time, then we have so called time optimal control problem where $L(x, u) = 1$.

An important class of perturbations are when for infinitesimal Δt, between time $(\tau - \Delta t)$ and τ, we delete the evolution $f(x, u)$. This leads to perturbation

$$\delta x(\tau) = -\begin{bmatrix} f(x, u) \\ L(x, u) \end{bmatrix} \Delta t.$$

We can also add the evolution $f(x, u)$, this leads to peturbation

$$\delta x(\tau) = \begin{bmatrix} f(x, u) \\ L(x, u) \end{bmatrix} \Delta t,$$

since $\lambda'(\tau)\delta x(\tau) > 0$, for both perturbations, we get

$$H(x(t), \lambda(t), u) = \lambda(t)' f(x(t), u) + L(x, u) = 0.$$

The control minimizes the Hamiltonian and the minimum value is zero.

Example 1 Consider the system $\ddot{x} = u$, we want to bring the system to origin $x, \dot{x} = 0$ in minimum time with the constraint $|u| \leq 1$. We can write the system in standard form as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}$$

Writing the Hamiltonian $H = \lambda_1 x_2 + \lambda_2 u$ with

$$(\dot{\lambda}_1, \dot{\lambda}_2) = -(0, \lambda_1),$$

then $\lambda_1 = c$ is a constant and $\lambda_2 = \lambda_0 + ct$. Then $u = -sgn(\lambda_2)$. We only have two controls ± 1. Let us study the trajectories under these two controls

$$\begin{aligned} \dot{x}_1 &= x_2 && (10) \\ \dot{x}_2 &= 1 && (11) \end{aligned}$$

Then the trajectories are $x_2(t) = x_2(0) + t$ and $x_1(t) = \frac{t^2}{2} + x_2(0)t + x_1(0)$. Then $x_1(t) = \frac{x_2(t)^2}{2} + c_1$. These trajectoris are sketched below in fig. 3 (bold).

$$\begin{aligned} \dot{x}_1 &= x_2 && (12) \\ \dot{x}_2 &= -1 && (13) \end{aligned}$$

Then the trajectories are $x_2(t) = x_2(0) - t$ and $x_1(t) = -\frac{t^2}{2} + x_2(0)t + x_1(0)$. Then $x_1(t) = -\frac{x_2(t)^2}{2} + c_2$. These trajectoris are sketched below in fig. 3 (dashed).

Since we can only switch one, then if we are above the arc AOB, then we follow dotted curve and come to bold curve and go to origin. If we are below the arc AOB, then we follow bold curve and come to dotted curve and go to origin. If we are on arc AOB, we donot switch.

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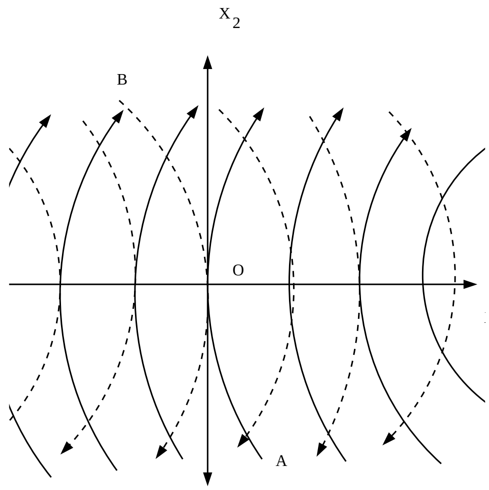

Figure 3: Figure shows optimal trajectories in Eq. 10 (in bold) and Eq. 12 (in dotted).

0.1 Transversality

Consider now a variant of the problem. Instead of reaching a final point x_f, we want to reach a surface/manifold $\mathcal{M}$. Let us say we have a optimal control u that reaches $\mathcal{M}$ optimally at point x_f. Then at x_f we have two vector spaces T_{x_f} and N_{x_f}, tangent and normal to $\mathcal{M}$. Let us choose a basis $e_1, \dots, e_m$ for N_{x_f}. We can decompose the perturbation $\delta x(T)$ as $\delta x_n(T)$ normal to M and $\delta x_t(T)$ parallel to $\mathcal{M}$. We can write $\delta x_n(T)$ in terms of $e_1, \dots, e_m$ as coordinates $(y_1, \dots, y_m)$. Let $y = (y_1, \dots, y_m, \delta x_{n+1})$. Note y forms a cone that doesnot include the point $p = (\underbrace{0, \dots 0}_{m}, -1)$ (the last cordinate is the cost coordinate) else we have a perturbation that is tangent to $\mathcal{M}$ and reduces the cost. Hence we can find $\mu = (\mu_1, \dots, \mu_m, \lambda_{n+1})$ that separates p from y cone. Then $\mu^T p < 0$ and $\mu^T y \geq 0$. Then $\lambda = \sum_{i=1}^m \mu_i e_i + \lambda_{n+1}(\underbrace{0, \dots, 0}_{n}, 1)'$. Then observe $\lambda^T \delta x(T) = \mu^T y \geq 0$. Furthermore, the system part of λ call λ_S is $\perp N_{x_f}$. This is called transversality.

Example 2 Consider the system $\ddot{x} = u$, we want to bring the system to origin $x = 0$ ($\dot{x}$ can be anything) in minimum time with the constraint $|u| \leq 1$. We can write the system in standard form as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}$$

As before writing the Hamiltonian $H = \lambda_1 x_2 + \lambda_2 u$ with

$$(\dot{\lambda}_1, \dot{\lambda}_2) = -(0, \lambda_1),$$

then $\lambda_1 = c$ is a constant and $\lambda_2 = \lambda_0 + ct$. Then $u = -sgn(\lambda_2)$. We only have two controls ± 1. We have already studied the trajectories under these two controls. Since at terminal time we donot have restriction on $\dot{x}$, we just want to reach the axis $x_1 = 0$. This means at terminal time $\lambda_2 = 0$ by transversality. That means sign of λ_2 never changes and we have no switching.

Then if we are above the arc AOB (or on arc OB), then we follow dotted curve and come to $x_1 = 0$ axis. If we are below the arc AOB (or on arc OA), then we follow bold curve and come to $x_1 = 0$ axis.

Example 3 Consider the nonholonomic integrator,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} u \\ v \\ xv - uy \end{bmatrix}.$$

Find the optimal control (u, v) that steers $(x(0), y(0), z(0)) = (0, 0, 0)$ to $(x(1), y(1), z(1)) = (0, 0, 1)$ and minimize

$$\eta = \frac{1}{2} \int_0^1 u^2 + v^2 dt$$

.

Since the final time is fixed, we can demand another state variable θ, such that $\dot{\theta} = 1$ and $\theta(t_f) = 1$, we call it the time coordinate and we also have the cost coordinate. Then using maximum principle

$$H = \lambda_1 u + \lambda_2 v + \lambda_3 (xv - yu) + \lambda_4 + \frac{u^2 + v^2}{2}$$

Minimizing the hamiltonian with u, v gives $u = -\lambda_1 + y\lambda_3$ and $v = -\lambda_2 - x\lambda_3$ and

$$\begin{aligned} \dot{\lambda}_1 &= -\lambda_3 v \\ \dot{\lambda}_2 &= \lambda_3 u \\ \dot{\lambda}_3 &= 0 \end{aligned}$$

$$\begin{aligned} \dot{u} &= 2\lambda_3 v \\ \dot{v} &= -2\lambda_3 u \end{aligned}$$

Then $u = A\cos(\omega t + \theta)$ and $v = A\sin(\omega t + \theta)$. Now integrating and matching boundary conditions we get $\omega = 2n\pi$ and $A = \omega$. For minimum cost $n = 1$. Optimal controls are sinusoids.

0.2 Maximum principle on Lie Groups

We now consider control systems of the form

$$\dot{x} = (\sum_i u_i \Omega_i)x, \quad x(0) = I$$

Where $\Omega_i \in \mathfrak{g}$ the Lie algebra of a Lie Group G. Let $\mathfrak{h} = span\{\Omega_i\} \subset \mathfrak{g}$.

We can write the above system as

$$\dot{x} = U(t)x, \quad x(0) = I, U(t) \in \mathfrak{h}$$

We want to steer the system to x_f and minimize cost $\int_0^{t_f} L(U)dt$. As before we add a cost coordinate

$$\dot{x}_{n+1} = L(U).$$

and we want to minimize $x_{n+1}(t_f)$.

We do a *needle peturbations* on controls. For infinitesimal Δt, between time $(\tau - \Delta t)$ and τ, we change the control from U to V. How does the final point change. Then observe $\delta x(\tau) = (V - U)x\Delta t$, then

$$\delta x(T) = \Theta \delta x(\tau) = \Theta(V - U)x(\tau)\Delta t = \underbrace{\Theta(V - U)\Theta^{-1}}_{\Delta\Omega} x(T)\Delta t$$

where Θ is the propagator from $x(\tau)$ to $x(T)$. Now note that $\Theta(V - U)\Theta^{-1} \in \mathfrak{g}$. Then

$(\Delta\Omega, \delta x_{n+1})$ forms a cone $\mathcal{C}$ and hence can be separates from $(0, -1)$, by multipliers (M, λ_{n+1}).

Therefore

$$\langle M, \Delta\Omega \rangle + \lambda_{n+1}\delta x_{n+1} \geq 0 \tag{14}$$

on the cone. As before we can choose $\lambda_{n+1} = 1$ when problem is normal.

Lets take the inner product

$$\langle M, \Delta\Omega \rangle = tr(M\Delta\Omega) = tr(M\Theta(V - U)\Theta^{-1}) = tr(\Theta^{-1}M\Theta(V - U))$$

Let $\Theta^{-1}M\Theta = M(\tau)$, then we have from Eq. (14), then note $\delta x_{n+1}(T) = \delta x_{n+1}(\tau)$

$$\begin{aligned} tr(M(\tau)(V-U))\Delta t + (L(V)-L(U))\Delta t &\geq 0 \\ tr(M(\tau)V) + L(V) &\geq tr(M(\tau)U) + L(U) \end{aligned}$$

Then optimal control U minimizes the Hamiltonian

$$H = tr(M(\tau)U) + L(U)$$

where $M(\tau) = \Theta(\tau)M(0)\Theta^{-1}(\tau)$, where $\Theta(\tau)$ is the propagator from 0 to τ.
Then

$$\dot{M} = [U, M],$$

this is the equation for the multipliers.

As simple example $G = SO(n)$ and $\mathfrak{g} = so(n)$. For $U, V \in so(n)$ skew symmetric matrices, we have a inner product $\langle U, V\rangle = tr(UV)$.

Example 4 We now consider the system on $SO(n)$,

$$\dot{x} = \Omega(t)x, \quad x(0) = I$$

Where $\Omega(t) \in so(n)$ the Lie algebra of a Lie Group G. We want to find the optimal $\Omega(t)$, to steer the system from I to x_f and minimize

$$\int_0^{t_f} \|\Omega(t)\| dt$$

Let $\frac{d\tau}{dt} = \|\Omega(t)\|$, then expresing the equation in terms of τ, we have

$$\frac{dx}{d\tau} = \underbrace{\frac{\Omega(t)}{\|\Omega(t)\|}}_{\Omega_1(t)} x, \quad x(0) = I$$

So the problem is simply steers

$$\dot{x} = \Omega_1(t)x, \quad x(0) = I, \|\Omega_1\| = 1$$

to x_f in minimum time.

Then we minimize the Hamiltonian for $M \in so(n)$

$$tr(\Omega_1(t)M(t)), \quad \dot{M} = [\Omega_1, M].$$

Then we allign $\Omega_1 = \alpha M$ and this gives $\dot{M} = 0$ and $\Omega_1 =$ constant. Then the trajectory is simply

$x(\tau) = \exp(\Omega_1 \tau)$ we choose a Ω_1 such that $\exp(\Omega_1 t_f) = x_f$.

These are called *geodesics*.

Example 5 Consider the control system on $SO(3)$ defined by

$$\dot{x} = (\underbrace{\cos\theta(t)\Omega_x + \sin\theta(t)\Omega_y}_{\Omega_1(t)})x,$$

where

$$\Omega_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Omega_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \Omega_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

We want to find optimal $\theta(t)$ to steer from $x(0) = I$ to $\exp(\Omega_z)$.

Then we minimize the Hamiltonian for $M \in so(3)$

$$tr(\Omega_1(t)M(t)), \quad \dot{M} = [\Omega_1, M].$$

We can write $M = m_x\Omega_x + m_y\Omega_y + m_z\Omega_z$. Again to minimize we choose

$$\Omega_1(t) = -(\frac{m_x}{\sqrt{m_x^2 + m_y^2}}\Omega_x + \frac{m_y}{\sqrt{m_x^2 + m_y^2}}\Omega_y)$$

i.e, allign Ω_1 with M. Then observe m_z never changes and is constant and we get

$$\begin{aligned} \dot{m}_x &= -\frac{m_y m_z}{\sqrt{m_x^2 + m_y^2}} \\ \dot{m}_y &= \frac{m_x m_z}{\sqrt{m_x^2 + m_y^2}} \end{aligned}$$

Then note $m_x^2 + m_y^2$ is constant and we can write the above equations as

$$\begin{aligned} \dot{m}_x &= -Cm_y \\ \dot{m}_y &= Cm_x \end{aligned}$$

Then $m_x = A\cos(Ct+\beta)$ and $m_y = A\sin(Ct+\theta)$ and $\Omega_1 = \cos(Ct+\beta')\Omega_x + \sin(Ct+\beta')\Omega_y$.

Let $y = \exp(-Ct\Omega_z)x$, then we get

$$\dot{y} = (C\Omega_z + \cos\beta'\Omega_x + \sin\beta'\Omega_y)y,$$

$$y(t) = \exp((C\Omega_z + \cos\beta'\Omega_x + \sin\beta'\Omega_y)t)$$

$$x(t) = \exp(-Ct\Omega_z)\exp((C\Omega_z + \cos\beta'\Omega_x + \sin\beta'\Omega_y)t)$$

We have to find C, β' such that we get to x_f in smallest time.

1 Excercises

1. For the system $\ddot{x} = u$, with $\|u\| \leq 1$, find the optimal way to steer the system from $(x(0), \dot{x}(0))$ to $(x, \dot{x}) = (0, 0)$ and minimize
$$\int_0^{t_f} \|u\| dt.$$

2. For the system $\ddot{x} = u$, with $\|u\| \leq 1$, find the optimal way to steer the system $(x(0), \dot{x}(0))$ to $x = 0$ and minimize
$$\int_0^{t_f} \|u\| dt.$$

3. For the system $\ddot{x} + x = u$, with $\|u\| \leq 1$, find the time optimal way to steer the system from $(x(0), \dot{x}(0))$ to $(x, \dot{x}) = (0, 0)$.

4. For the system $\ddot{x} + x = u$, with $\|u\| \leq 1$, find the time optimal way to steer the system $(x(0), \dot{x}(0))$ to $x = 0$.

5. For the system $\ddot{x} + x = u$, with $\|u\| \leq 1$, find the optimal way to steer the system from $(x(0), \dot{x}(0))$ to $(x, \dot{x}) = (0, 0)$ and minimize
$$\int_0^{t_f} \|u\| dt.$$

6. For the system $\ddot{x} + x = u$, with $\|u\| \leq 1$, find the optimal way to steer the system $(x(0), \dot{x}(0))$ to $x = 0$ and minimize
$$\int_0^{t_f} \|u\| dt.$$

7. Consider the control system on $SO(3)$ defined by

$$\dot{x} = (u\Omega_x + v\Omega_y)x,$$

find optimal (u, v) to steer from $x(0) = I$ to $x(1) = \exp(\Omega_z)$ and minimize $\int_0^1 u^2 + v^2 dt$.

8. Steer $\dot{x} = u$ form $x(0)$ to $x(1)$ and minimize $\int_0^1 x^2 + u^2 dt$.

9. Steer $\dot{x} = -x + u$ form $x(0)$ to $x(1)$ and minimize $\int_0^1 x^2 + u^2 dt$.

10. Consider the control system on $SO(3)$ defined by

$$\dot{x} = (u\Omega_x + v\Omega_y)x,$$

find optimal (u, v) to steer from $x(0) = I$ to $\exp(\Omega_z)$ and minimize $\int \|u\| + \|v\| dt$.