# Chapter3: Optimal Control

Until now, our focus has been on controllability. Showing we can steer our control system between points of interest. In this chapter we turn to another important question in control. How to optimally steer a dynamical system. In this chapter we will learn about Pontryagin's maximum principle.

Consider the control system

$$\dot{x} = f(x, u), \quad x \in R^n, u \in \Omega \subset R^m$$

We want to steer the system from $x_0$ to $x_f$ and want to minimize

$$\eta = \int L(x, u) dt,$$

We can write this as

$$\begin{aligned} \dot{x} &= f(x, u) && (1) \\ \dot{x}_{n+1} &= L(x, u) && (2) \end{aligned}$$

and say we want to minimize $x_{n+1}$ at final time which say is $T$ with a control $u$ which is optimal. We can make $x$, $n+1$ dimensional, $(x, x_{n+1})$ and write the above system as

$$\dot{x} = f(x, u) \tag{3}$$

with new $f$.

We do a *needle peturbations* on controls. For infinitesimal $\Delta t$, between time $(\tau - \Delta t)$ and $\tau$, we change the control from $u$ to $v$. How does the final point change. Then observe

$$\begin{aligned} \delta x(\tau) &= (f(x, v) - f(x, u))\Delta t && (4) \\ \delta x(T) &= \Phi(T, \tau)\delta x(\tau) && (5) \\ \dot{\Phi}(t, \tau) &= \underbrace{\frac{\partial f}{\partial x}|_{(x^*(t), u(t))}}_{A(t)} \Phi(t, \tau) && (6) \end{aligned}$$

Observe $\delta x(T)$ lies in a cone. If I perturb at $\tau_1$ and $\tau_2$, I get end point perturbation $\delta_1 x(T)$ and $\delta_2 x(T)$. Then by perturbing simulataneously for time $\alpha \Delta t$ and $(1-\alpha)\Delta t$, we get

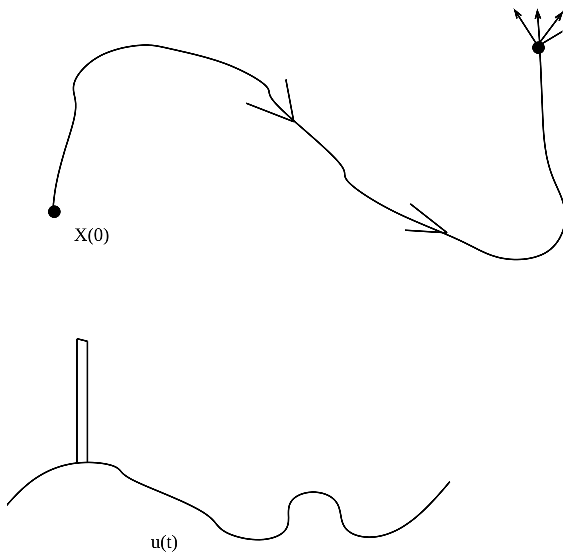


Figure 1: Figure a shows the trajectory from $x_0$ to $x_f$ and how end point is perturbed as we make a needle perturbation to control in Fig. b.

$\delta x(T) = \alpha \delta_1 x(T) + (1-\alpha)\delta_2 x(T)$. Hence any convex combination is achievable. End point perturbations lie in a convex cone $\mathcal{C}$. Now we donot want that the point $p = (\underbrace{0, \ldots, 0}_{n}, -1) \notin \mathcal{C}$, else we make a perturbation that fixes $x(T)$ and decreases $x_{n+1}$, then how is out trajectory optimal. Then we can find a hyperplane which separates $p$ from $\bar{\mathcal{C}}$, (first assume $p \notin \bar{\mathcal{C}}$. Then there are multipliers $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$, such that $\lambda^T p < 0$ and $\lambda^T \bar{\mathcal{C}} \geq 0$. It means $\lambda_{n+1} > 0$, which can be chosen as 1 and $\lambda^T \delta x(T) \geq 0$

$$\lambda' \Phi(T, \tau) \delta x(\tau) \geq 0; \quad (\Phi(T, \tau)' \lambda)' \delta x(\tau) \geq 0$$

call $\Phi(T, \tau)' \lambda = \lambda(\tau)$, then observe

$$\dot{\lambda}(\sigma) = -A^T(\sigma) \lambda.$$

$$\lambda'(\tau)(f(x(\tau), v) - f(x(\tau), u)) \geq 0$$

Note by definition of $A(t)$ in 4, the last row of $A^T$ is zero. Hence $\lambda_{n+1} = 1$ throughout. Then define

$$H(x(t), \lambda(t), u) = \lambda(t)' f(x, u),$$

and

$$
\begin{aligned}
H(x(t), \lambda(t), u) &\leq H(x(t), \lambda(t), v), \quad \forall v && (7)\\
\dot{\lambda}' &= -\lambda' \frac{\partial H}{\partial x} && (8)\\
\dot{x} &= (\frac{\partial H}{\partial \lambda})' && (9)
\end{aligned}
$$

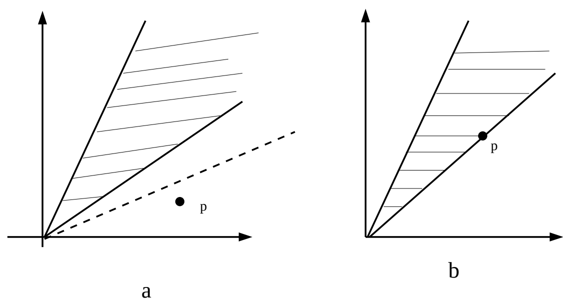


Figure 2: Figure a shows the end point cone and point $p$ outside it which can be separated by a hyperplane. Figure b shows when $p$ lie on boundary of $\bar{\mathcal{C}}$.

Since $\lambda_{n+1} = 1$, we can write

$$H(x(t), \lambda(t), u) = \lambda(t)' f(x(t), u) + L(x, u),$$

where $\lambda, f$ refers to first $n$ coordinates.

Observe, we assumed $p \notin \bar{\mathcal{C}}$. Such problems are called *normal problems*. It is possible that $p \in \partial\bar{\mathcal{C}}$, just on the boundary. Then recall $\lambda^T p = 0$, separating plane passes through $p$. Then since $p = (0, \ldots, 0, -1)$, we have $x_{n+1} = 0$ and then

$$H(x(t), \lambda(t), u) = \lambda(t)' f(x(t), u).$$

Such problems are called *abnormal*. When we are interested in minimizing time, then we have so called time optimal control problem where $L(x, u) = 1$.

An important class of perturbations are when for infinitesimal $\Delta t$, between time $(\tau - \Delta t)$ and $\tau$, we delete the evolution $f(x, u)$. This leads to perturbation

$$\delta x(\tau) = -\left[ \begin{array}{c} f(x, u) \\ L(x, u) \end{array} \right] \Delta t.$$

We can also add the evolution $f(x, u)$, this leads to peturbation

$$\delta x(\tau) = \left[ \begin{array}{c} f(x, u) \\ L(x, u) \end{array} \right] \Delta t,$$

since $\lambda'(\tau)\delta x(\tau) > 0$, for both perturbations, we get

$$H(x(t), \lambda(t), u) = \lambda(t)' f(x(t), u) + L(x, u) = 0.$$

The control minimizes the Hamiltonian and the minimum value is zero.

**Example 1** Consider the system $\ddot{x} = u$, we want to bring the system to origin $x, \dot{x} = 0$ in minimum time with the constraint $|u| \leq 1$. We can write the system in standard form as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}$$

Writing the Hamiltonian $H = \lambda_1 x_2 + \lambda_2 u$ with

$$(\dot{\lambda}_1, \dot{\lambda}_2) = -(0, \lambda_1),$$

then $\lambda_1 = c$ is a constant and $\lambda_2 = \lambda_0 + ct$. Then $u = -sgn(\lambda_2)$. We only have two controls $\pm 1$. Let us study the trajectories under these two controls

$$\begin{aligned} \dot{x}_1 &= x_2 &\quad (10) \\ \dot{x}_2 &= 1 &\quad (11) \end{aligned}$$

Then the trajectories are $x_2(t) = x_2(0) + t$ and $x_1(t) = \frac{t^2}{2} + x_2(0)t + x_1(0)$. Then $x_1(t) = \frac{x_2(t)^2}{2} + c_1$. These trajectoris are sketched below in fig. 3 (bold).

$$\begin{aligned} \dot{x}_1 &= x_2 &\quad (12) \\ \dot{x}_2 &= -1 &\quad (13) \end{aligned}$$

Then the trajectories are $x_2(t) = x_2(0) - t$ and $x_1(t) = -\frac{t^2}{2} + x_2(0)t + x_1(0)$. Then $x_1(t) = -\frac{x_2(t)^2}{2} + c_2$. These trajectoris are sketched below in fig. 3 (dashed).

Since we can only switch one, then if we are above the arc AOB, then we follow dotted curve and come to bold curve and go to origin. If we are below the arc AOB, then we follow bold curve and come to dotted curve and go to origin. If we are on arc AOB, we donot switch.

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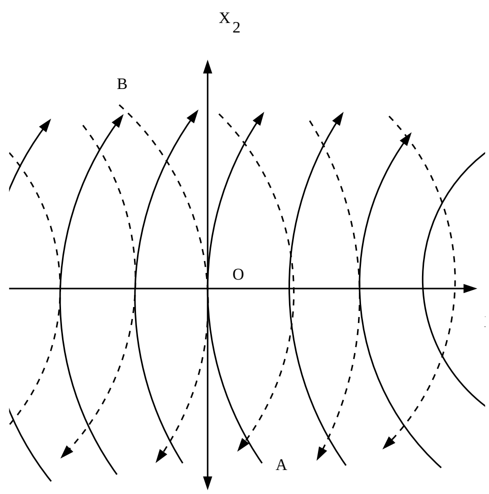


Figure 3: Figure shows optimal trajectories in Eq. 10 (in bold) and Eq. 12 (in dotted).

### 0.1 Transversality

Consider now a variant of the problem. Instead of reaching a final point $x_f$, we want to reach a surface/manifold $\mathcal{M}$. Let us say we have a optimal control $u$ that reaches $\mathcal{M}$ optimally at point $x_f$. Then at $x_f$ we have two vector spaces $T_{x_f}$ and $N_{x_f}$, tangent and normal to $\mathcal{M}$. Let us choose a basis $e_1, \dots, e_m$ for $N_{x_f}$. We can decompose the perturbation $\delta x(T)$ as $\delta x_n(T)$ normal to $M$ and $\delta x_t(T)$ parallel to $\mathcal{M}$. We can write $\delta x_n(T)$ in terms of $e_1, \dots, e_m$ as coordinates $(y_1, \dots, y_m)$. Let $y = (y_1, \dots, y_m, \delta x_{n+1})$. Note $y$ forms a cone that doesnot include the point $p = (\underbrace{0, \dots 0}_{m}, -1)$ ( the last cordinate is the cost coordinate) else we have a perturbation that is tangent to $\mathcal{M}$ and reduces the cost. Hence we can find $\mu = (\mu_1, \dots, \mu_m, \lambda_{n+1})$ that separates $p$ from $y$ cone. Then $\mu^T p < 0$ and $\mu^T y \geq 0$. Then $\lambda = \sum_{i=1}^m \mu_i e_i + \lambda_{n+1}(\underbrace{0, \dots, 0}_{n}, 1)'$ . Then observe $\lambda^T \delta x(T) = \mu^T y \geq 0$. Furthermore, the system part of $\lambda$ call $\lambda_S$ is $\perp N_{x_f}$. This is called transversality.

**Example 2** Consider the system $\ddot{x} = u$, we want to bring the system to origin $x = 0$ ($\dot{x}$ can be anything) in minimum time with the constraint $|u| \leq 1$. We can write the system in standard form as

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u
\end{aligned}
$$

As before writing the Hamiltonian $H = \lambda_1 x_2 + \lambda_2 u$ with

$$(\dot{\lambda}_1, \dot{\lambda}_2) = -(0, \lambda_1),$$

then $\lambda_1 = c$ is a constant and $\lambda_2 = \lambda_0 + ct$. Then $u = -sgn(\lambda_2)$. We only have two controls $\pm 1$. We have already studied the trajectories under these two controls. Since at terminal time we donot have restriction on $\dot{x}$, we just want to reach the axis $x_1 = 0$. This means at terminal time $\lambda_2 = 0$ by transversality. That means sign of $\lambda_2$ never changes and we have no switching.

Then if we are above the arc AOB (or on arc OB), then we follow dotted curve and come to $x_1 = 0$ axis. If we are below the arc AOB (or on arc OA), then we follow bold curve and come to $x_1 = 0$ axis.

**Example 3** Consider the nonholonomic integrator,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} u \\ v \\ xv - uy \end{bmatrix}.$$

Find the optimal control $(u, v)$ that steers $(x(0), y(0), z(0)) = (0, 0, 0)$ to $(x(1), y(1), z(1)) = (0, 0, 1)$ and minimize

$$\eta = \frac{1}{2} \int_0^1 u^2 + v^2 dt$$

.

Since the final time is fixed, we can demand another state variable $\theta$, such that $\dot{\theta} = 1$ and $\theta(t_f) = 1$, we call it the time coordinate and we also have the cost coordinate. Then using maximum principle

$$H = \lambda_1 u + \lambda_2 v + \lambda_3 (xv - yu) + \lambda_4 + \frac{u^2 + v^2}{2}$$

Minimizing the hamiltonian with $u, v$ gives $u = -\lambda_1 + y\lambda_3$ and $v = -\lambda_2 - x\lambda_3$ and

$$\begin{aligned} \dot{\lambda}_1 &= -\lambda_3 v \\ \dot{\lambda}_2 &= \lambda_3 u \\ \dot{\lambda}_3 &= 0 \end{aligned}$$

$$\begin{aligned} \dot{u} &= 2\lambda_3 v \\ \dot{v} &= -2\lambda_3 u \end{aligned}$$

Then $u = A\cos(\omega t + \theta)$ and $v = A\sin(\omega t + \theta)$. Now integrating and matching boundary conditions we get $\omega = 2n\pi$ and $A = \omega$. For minimum cost $n = 1$. Optimal controls are sinusoids.

# 1 Excercises

1. For the system $\ddot{x} = u$, with $\|u\| \leq 1$, find the optimal way to steer the system from $(x(0), \dot{x}(0))$ to $(x, \dot{x}) = (0, 0)$ and minimize

$$\int_0^{t_f} \|u\| dt.$$

2. For the system $\ddot{x} = u$, with $\|u\| \leq 1$, find the optimal way to steer the system $(x(0), \dot{x}(0))$ to $x = 0$ and minimize

$$\int_0^{t_f} \|u\| dt.$$

3. For the system $\ddot{x} + x = u$, with $\|u\| \leq 1$, find the time optimal way to steer the system from $(x(0), \dot{x}(0))$ to $(x, \dot{x}) = (0, 0)$.

4. For the system $\ddot{x} + x = u$, with $\|u\| \leq 1$, find the time optimal way to steer the system $(x(0), \dot{x}(0))$ to $x = 0$.

5. For the system $\ddot{x} + x = u$, with $\|u\| \leq 1$, find the time optimal way to steer the system from $(x(0), \dot{x}(0))$ to $\dot{x} = 0$.

6. For the system $\ddot{x} = u$, with $\|u\| \leq 1$, find the time optimal way to steer the system $(x(0), \dot{x}(0))$ to $\dot{x} = 0$.

7. For the system $\ddot{x} + x = u$, with $\|u\| \leq 1$, find the optimal way to steer the system from $(x(0), \dot{x}(0))$ to $(x, \dot{x}) = (0, 0)$ and minimize

$$\int_0^{t_f} \|u\| dt.$$

8. For the system $\ddot{x} + x = u$, with $\|u\| \leq 1$, find the optimal way to steer the system $(x(0), \dot{x}(0))$ to $x = 0$ and minimize

$$\int_0^{t_f} \|u\| dt.$$

9. Steer $\dot{x} = u$ form $x(0)$ to $x(1)$ and minimize $\int_0^1 x^2 + u^2 dt$.
10. Steer $\dot{x} = -x + u$ form $x(0)$ to $x(1)$ and minimize $\int_0^1 x^2 + u^2 dt$.