Computational Issues in Nonlinear Dynamics and Control

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Typical Problems

• Numerical Computation of Invariant Manifolds
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- Numerical Computation of Invariant Manifolds
- Numerical Solution of Optimal Control Problems
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An Important Problem

Infinite Horizon Optimal Control

$$\min_{u(0:\infty)} \int_{0}^{\infty} l(x, u) \, dt$$

$$\dot{x} = f(x, u), \quad x(0) = x^0$$

$$x \in \mathbb{IR}^{n \times 1}, \quad u \in \mathbb{IR}^{m \times 1}$$
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Optimal Cost and Optimal Feedback

\[ \pi(x^0) = \min_{u(0:\infty)} \int_{0}^{\infty} l(x, u) \, dt, \quad u^*(0) = \kappa(x^0) \]
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Hamiltonian, a function of \( x, u \) and a new variable \( p \in \mathbb{R}^{1 \times n} \)

\[ \mathcal{H}(p, x, u) = pf(x, u) + l(x, u) \]
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\min_{u(0:\infty)} \int_{0}^{\infty} l(x, u) \ dt
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Hamilton Jacobi Bellman Equations

\[
0 = \min_{u} \mathcal{H}\left(\frac{\partial \pi}{\partial x}(x), x, u\right)
\]

\[
\kappa(x) = \arg\min_{u} \mathcal{H}\left(\frac{\partial \pi}{\partial x}(x), x, u\right)
\]
Problem: Find a feedback $u = \kappa(x)$ so that the closed loop system is (locally) asymptotically stable around $x = 0$.

Solution: Choose a Lagrangian $l(x, u) \geq 0$ and solve the infinite horizon optimal control problem. Under suitable conditions the optimal feedback $u = \kappa(x)$ is stabilizing on some domain around $x = 0$ and this can be verified because the optimal cost $\pi(x) \geq 0$ is a Lyapunov function,

$$\frac{d}{dt} \pi(x(t)) = \frac{\partial \pi}{\partial x}(x(t)) f(x(t), \kappa(x(t))) = -l(x(t), \kappa(x(t))) \leq 0$$
Classic Example: LQR

\[ f(x, u) = Fx + Gu, \quad l(x, u) = \frac{1}{2} (x'QX + u'Ru) \]
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Optimal Cost and Optimal Feedback

\[ \pi(x) = \frac{1}{2} x'Px, \quad \kappa(x) = Kx \]

The HJB equations reduce to a quadratic (algebraic Riccati) equation and a linear equation

\[ 0 = F'P + PF + Q - PGR^{-1}G'P, \quad K = -R^{-1}G'P \]
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Theorem: If \( Q \geq 0, \ R > 0, \ (F, G) \) stabilizable and \( (Q^{1/2}, F) \) detectable then there exist a unique nonnegative definite solution \( P \) to the Riccati equation and the feedback \( u = Kx \) is asymptotically stable, i.e., all the poles of \( F + GK \) are in the open left half plane.
Another Important Problem

Finite Horizon Optimal Control

\[
\min_{u(0:T)} \int_0^T l(t, x, u) \, dt + \pi^T(x(T))
\]

\[
\dot{x} = f(t, x, u)
\]

\[
0 = g(x(0), X(T))
\]

\[
u(t) \in \mathcal{U}(t, x)
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Pontryagin Maximum Principle:
If \(x^*(0 : T), u^*(0 : T)\) is optimal then there exists \(p : [0, T] \rightarrow IR^{1 \times n}\) such that

\[
\dot{x}^*_i = \frac{\partial \mathcal{H}}{\partial p_i}(t, p, x^*, u^*)
\]

\[
\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial x_i}(t, p, x^*, u^*)
\]

\[
u^* = \arg\min_{u \in U(T, x^*)} \mathcal{H}(t, p, x^*, u^*)
\]

\[
\mathcal{H}(t, p, x, u) = pf(t, x, u) + l(t, x, u)
\]
Analyze vs Discretize

There is a PMP for infinite horizon OCP and an HJB for finite horizon OCP but in the interests of time we shall not discuss them.
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In order to solve these problems we have to discretize them. Discretize the optimal control problem or discretize the HJB or PMP equations?
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- first analyze and then discretize
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• first analyze and then discretize
• first discretize and then analyze.
Commutative Diagrams?

Infinite Horizon OCP $\rightarrow$ Dynamic Program

↓ ↓

HJB PDE $\rightarrow$ Dynamic Programming Equation
Infinite Horizon OCP $\rightarrow$ Dynamic Program
\[ \downarrow \]
HJB PDE $\rightarrow$ Dynamic Programming Equation

Finite Horizon OCP $\rightarrow$ Nonlinear Program
\[ \downarrow \]
PMP $\rightarrow$ Karush Kuhn Tucker Conditions
Discretization of the HJB equation

For simplicity of exposition assume \( n = 2, \, m = 1 \). Choose a rectangle around \( x = 0 \) and partition it with stepsize \( h \). Let \( x^{i,j} \) denote the \( i, j \) node. Let \( \pi^{i,j} \) be the current computed approximation to the optimal cost at the \( x^{i,j} \).

For each \( i,j \) solve for the next approximation \( \kappa^{i,j} \) to the optimal feedback

\[
\kappa^{i,j} = \arg\min_u \left\{ (\pi^{i+1,j} - \pi^{i-1,j}, \pi^{i,j+1} - \pi^{i,j-1}) f(x^{i,j}, u) + 2hl(x^{i,j}, u) \right\}
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The boundary condition is

\( \bar{\pi}^{i_0,j_0} = 0 \) where \( x^{i_0,j_0} = 0 \).

This is called policy iteration and it is not very efficient because it sweeps through the nodes many times.
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Discretization of the Optimal Control Problem

Instead we can discretize the optimal control problem. Assume that the control takes on discrete values $u_k$ and time is measured in steps of $h$. 

Define $\bar{f}(x_{i,j}, u_k)$ to be the state node closest to $x_{i,j} + f(x_{i,j}, u_k)h$. Then on the state and control grids we have the discrete dynamics:

$$x_{i,j}^{t+1} = \bar{f}(x_{i,j}, u_k)$$

$x_{i,j}(0) = x_{i,j}$ and we minimize by choice of control sequence $\pi_{i,j} = \min_{u(0:\infty)} \sum_{t=0:h: \infty} l(x,u)h \kappa_{i,j} = u^*(0)$.
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Discretization of the Optimal Control Problem

Dynamic Programming Equation (DPE)

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\pi^{i,j} = \min_{u_k} \left\{ l(x^{i,j}, u^k)h + \pi(\bar{f}(x^{i,j}, u^k)) \right\}
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This can be solved by policy iteration. Given the current approximation \(\pi(x^{i,j})\) to the optimal cost at grid points \(x^{i,j}\), define the next approximation to the optimal feedback as

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Approximation by a Markov Chain

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Partition the state space into a grid with spacing $h$ and partition time with spacing $k$. Construct a Controlled Markov Chain with transition probability $p(x^1|x^0, u)$ from gridpoint $x^0$ to grid point $x^1$ with control $u$. Choose a search radius $r$ and define

$$p(x^1|x^0, u) = \frac{\exp\left(-\left(\|x^1 - x^0 - (f(x^0, u)k\|^2\right)\right)}{\rho(x^0, u)}$$

$$\rho(x^0, u) = \sum_j \exp\left(-\left(\|x^1 - x^0 - (f(x^0, u)k\|^2\right)\right)$$

where the sum is over all gridpoints $x^j$ such that

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where the sum is over all gridpoints $x^j$ such that

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The cost is defined to be the expected value of

$$\sum_{t=0}^{\infty} l(x, u)$$
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In dimensions \( n = 2, 3 \) and \( m = 1, 2 \) this is a feasible method.

Boue and Dupuis have proven that a more sophisticated version converges to the true solution as \( h, k \) go to 0.
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Boue and Dupuis have proven that a more sophisticated version converges to the true solution as $h, k$ go to 0.

But in higher dimensions it difficult to implement.
Suppose that the speed of propagation $c(x) > 0$ through a medium varies with location. Consider any path $x(t)$ between the source $x^0 = 0$ and $x^1$. Then the propagation time along this path is $\int_0^t 1 \, d\tau$ so the Lagrangian $l(x, u) = 1$. 

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$$0 = \min_{\|u\| = 1} \left\{ \frac{\partial \pi}{\partial x}(x)c(x)u + 1 \right\}$$

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Fast Marching Method for the Eikonal Equation

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This method is due to Tsitsiklis and it was refined by Sethian, Falcone and others.

Partition the nodes into three families called accepted, narrow band and far. Initially the only node in accepted family is the origin and $\pi^{i_0,j_0} = 0$. 
Assume that $\pi^{i,j}$ has been computed for all the accepted nodes. For each node $x^{i,j}$ in the narrow band compute the rectilinear path to a node $x^{r,s}$ in the accepted region that minimizes the sum of travel time along the path plus $\pi^{r,s}$. This is typically done using Dykstra's method for finding the shortest path on a graph. Then accept the narrow band node that minimizes this sum. Repeat until the accepted nodes cover the region where the solution is desired. Different implementations of FMM use different ways of computing the sum of travel time along the path plus $\pi^{r,s}$. The advantage of FMM is that the computation visits each node much less often than in the naive approach. The FMM has been generalized to other optimal control problems but computing the minimum sums is more complicated because not every rectilinear path is feasible.
Fast Marching Method for the Eikonal Equation

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Consider trying to apply a grid based method. For the solution to be reasonably accurate we would need a substantial number of grid points in each coordinate direction, e.g., $10^2$. Then the total number of grid points is $10^{12}$ for attitude control and $10^{24}$ for position and attitude control. If we can process $100$ nodes a second that works out to about $300$ years for attitude control and $3 \cdot 10^{14}$ years for position and attitude control.
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HJB Equations and Conservation Laws

Suppose we have a time varying problem of the form

\[
\dot{x} = f(t, x) + g(t, x)u
\]

\[
l(t, x, u) = q(t, x) + \frac{1}{2}u'R(t, x)u
\]

Then the HJB PDEs reduce to the HJ PDE

\[
0 = \frac{\partial \pi}{\partial t} + \frac{\partial \pi}{\partial x} f - \frac{1}{2} \frac{\partial \pi}{\partial x} gRg' \left( \frac{\partial \pi}{\partial x} \right) + q
\]

Let \( p = \frac{\partial \pi}{\partial x} \) and take the Jacobian of the HJ equation to obtain the conservation law

\[
0 = \frac{\partial p}{\partial t} + \frac{\partial}{\partial x} F(t, x, p)
\]

where the flux term is

\[
F(t, x, p) = pf - \frac{1}{2}pgRg'p' + q
\]
HJB Equations and Conservation Laws

This connection has been used to take advantage of the highly developed methods for conservation laws to solve HJ and HJB equations.

\[ \pi(x) \in \mathbb{R}, \ p(x) \in \mathbb{R}^{1 \times n} \]

Another is that we are looking for a solution of the conservation law that is a closed one form,

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Invariant Manifold Methods

Hamilton Differential Equations

\[ \dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i} \]
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So we can compute \( \pi(x) \) by computing the stable manifold of the Hamiltonian dynamics.
Figure 1: Sketch of the balanced planar pendulum on a moving cart.
Figure 2: Each point is colored according to how high the cost of getting to the origin using this point as initial condition. The cost increases as the color changes from blue to red.
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These operations are commutative, associative and distributive,

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There is also a max-plus semiring where

\[ \xi \oplus \zeta = \max\{\xi, \zeta\} \]
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Consider the semigroup acting on function $\psi(x)$ for $T \geq 0$

$$
S_T(\phi)(x^T) = \min_{u(-T:0)} \left\{ \int_{-T}^{0} l(x(t), u(t)) \, dt + \phi(x(0)) \right\}
$$

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But is linear in the min-plus sense and $\pi(x)$ is an eigenvector corresponding to the eigenvalue 0 which is the $\otimes$ identity.

$$0 \otimes \pi(x) = S_T(\pi(x))$$
Min-Plus Methods

The power method is the standard way to find an eigenvector.

\[
\phi(x) \text{ and compute } \pi(x) = \lim_{N \to \infty} S^T \circ S^T \circ \cdots \circ S^T (\phi(x))
\]

where \(N\) is the number of composition factors.

To make the calculation finite dimensional \(\pi(x)\) is chosen as a min-plus combination of basis functions.

\[
\pi(x) = (\alpha_1 \otimes \psi_1(x)) \oplus \cdots \oplus (\alpha_k \otimes \psi_k(x))
\]

and a projection is done after each application of the semigroup.

This is very similar to policy iteration, the principle difference is the restriction to min-plus combinations of basis functions.

The number of basis functions needed for a given accuracy is exponential in the state dimension \(n\) but it is probably grows slower than the number of grid points.
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It also suffers from a curse of complexity as it requires computing the pointwise maxima (or minima) of a large number of functions which can be expensive.
Higher Order Methods

Why go to higher order methods? Here is a simple answer.
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Assuming the order constants are about the same size then to achieve the same accuracy $h_1^2 \approx h_3^4$ or $h_3 \approx \sqrt{h_1}$. 

If $h_1 = 0.01$ then $h_3 \approx 0.1$ so the number of grid points that is needed for a given accuracy is reduced by a factor of 10 in each dimension. If the state dimension is $n$ then the reduction in grid points is by a factor of $10^n$.

If the third order method takes $k_3(n)$ times longer to compute for each node then the reduction in computational time is by the factor $10^n k_3(n)$. Typically $k_3(n)$ is polynomial in $n$. 

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There are diminishing returns as we go to higher orders.

Consider a fifth order method with step size $h_5$. Then for same level of accuracy

$$h_1^2 \approx h_5^6 \quad \text{so} \quad h_5 \approx h_1^{1/3}$$

$$h_5 \approx 0.2154 \quad \text{when} \quad h_1 = 0.01$$
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Suppose the fifth order method takes $k_5(n)$ times longer for each node. Typically $k_d(n)$ grows exponentially in $d$. 
Richardson Extrapolation

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Suppose we have a first order method $M_1(h)$ for solving a problem using stepsize $h$. If the problem and the method are smooth enough then we expect that the error is a power series in $h$ with lowest order term a constant times $h^2$. Let $\alpha$ denote the true solution then with steps sizes $h$ and $2h$

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\alpha = M_1(h) + \beta_2 h^2 + \beta_3 h^3 + \ldots
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Szpiro and Dupuis have applied this technique to HJB equations.
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\end{align*}
\]

Multiply the first by $4/3$ and the second by $-1/3$ and add,

\[
\alpha = M_2(h) + O(h)^3
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where $\begin{align*}
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Singular PDEs

A first order quasilinear PDE

\[ 0 = \frac{\partial \phi}{\partial x}(x)a(x) + b(x, \phi(x)) \]

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We expand in power series.

\[
\begin{align*}
a(x) &= Ax + a^{[2]}(x) + a^{[3]}(x) + \ldots \\
b(x, \phi(x)) &= Cx + B\phi(x) + (b(x, \phi(x)))^{[2]} + (b(x, \phi(x)))^{[3]} + \ldots \\
\phi(x) &= Tx + \phi^{[2]}(x) + \phi^{[3]}(x) + \ldots
\end{align*}
\]

where \((\cdot)^{[d]}\) denotes terms homogeneous of degree \( d \).
Collect terms of first degree.

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The higher degrees terms can be found in a similar fashion.
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Many of the most important PDEs of nonlinear dynamics and control are essentially singular first order quasilinear including:

- PDEs for Stable, Unstable, Center, etc. Manifolds
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In discrete time the degree two nonresonance conditions are

\[ \alpha_{i_1} \alpha_{i_2} \neq \beta_j \]
Al’brecht’s Method

Al’brecht developed the power series method for HJB equations for the optimal cost and optimal feedback,

\[ \pi(x) = \frac{1}{2} x'Px + \pi^3(x) + \pi^4(x) + \ldots \]
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If the standard LQR conditions are satisfied then the Riccati equation has an unique nonnegative definite solution \( P \) and the linear part of the closed loop dynamics

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This guarantees there are no resonances so the higher degree terms of \( \pi, \kappa \) can be found by solving invertible linear equations.
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This method has been implemented in the Nonlinear Systems Toolbox.
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The HJB equations can be solved to degree $4$ in $\pi(x)$ and degree $3$ in $\kappa(x)$ for systems with state dimension $n = 25$ and control dimension $m = 8$ on a lap top.
Pros and Cons of Power Series Methods

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Going to higher degree approximations to $\pi(x)$ and $\kappa(x)$ increases their accuracy near $x = 0$.

Going to higher degree approximations can enlarge the basin of stability of the closed loop system but it is not guaranteed. It can also decrease it.

Going to higher degree approximations requires more memory. There are $n^d - 1$ choose $d$ monomials of degree $d$ in $n$ variables, approximately $n^d / d!$. 
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Patchy Methods

Figure: Optimal Cost of Inverting a Pendulum by a Torque at its Axis
Sequence of Patches

$\mathbf{x}_0 = 0$

$A$

$\mathbf{x}_1$

$\mathbf{x}_2$

$\mathbf{x}_0 = 0$

Figure: Sequence of Patches
The HJB equations are not singular away from the origin. The map

\[ \pi^{[d+1]} \mapsto \frac{\partial \pi^{[d+1]}}{\partial x}(x) f(x^1, u^1) \]

takes a polynomial of degree \( d + 1 \) to a polynomial of degree \( d \).
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Under suitable assumptions there is one positive root and one negative root. We take the positive root.
Invert a Pendulum

Figure: Periodicity of the Optimal Cost

The left axis is $-15 \leq \dot{\theta} \leq 15$ and the right axis is $-15 \leq \theta \leq 15$. From points on the ridges there are two optimal trajectories, one going to the left well and the other going to the right well.
Adaptive Algorithm

The algorithm is adaptive. It splits a patch in two when the relative residue of the first HJB equation is too high at the lower corners of a patch. It also lowers the upper level of a ring of patches if the relative residue is too high on it.

<table>
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<tr>
<th>Ring</th>
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<th>4</th>
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<tr>
<td>Initial Patch Level</td>
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The initial levels of the optimal cost were set at

\[(0.8)^2 \quad (1.1)^2 \quad (1.4)^2 \quad \ldots \quad (10.7)^2\]

Only the first ten patch levels were adjusted down.
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The last ring (34) contains 78 patches.
Figure: Patchy Optimal Cost to Level Set 70
Patchy Hauser Osinga Pendulum

Figure: Patchy Optimal Cost to Level Set 140
Patchy Hauser Osinga Pendulum

Figure: Patchy Optimal Cost to Level Set 210
Figure: Patchy Optimal Cost to Level Set 280
Figure: Patchy Optimal Cost to Level Set 350
Patchy Hauser Osinga Pendulum

Figure: Patchy Optimal Cost to Level Set 420
Figure: Patchy Optimal Cost to Level Set 490
Figure: Patchy Optimal Cost to Level Set 566
Error Comparison

A nonlinear change of state coordinates on an LQR problem yields a nonlinear optimal control problem.

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This shows that the patchy method can be very accurate and it is parallelizable.
A nonlinear change of state coordinates on an LQR problem yields a nonlinear optimal control problem.

The exact solution to the nonlinear problem is given by applying the nonlinear change of coordinates to the LQR solution.
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Three Dimensional Example

Here is a level set of the patchy method applied to a three dimensional problem

Figure: Level Set 55
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![Level Set 55](image)

Figure : Level Set 55

The complexity of keeping track of the patches probably makes the patchy method infeasible in higher dimensions.
There are feasible methods for solving HJB or DP equations in dimensions $n = 2$ or $n = 3$. It is questionable whether any of these methods are feasible when $n = 4$ or $n = 5$. It is unlikely that any of these methods are feasible when $n \geq 6$. Similar statements are true for the other PDEs of nonlinear control.
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Similar statements are true for the other PDEs of nonlinear control.
Typical Problem

\[
\min_{u(0:T)} \int_0^T l(x, u) \, dt.
\]

subject to

\[
\dot{x} = f(x, u), \quad 0 \leq g(x, u),
\]

\[
x(0) = x^0, \quad x(T) = x^T.
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\[
\dot{x}_i = \frac{\partial \mathcal{H}}{\partial p_i}(p, x, u^*)
\]

\[
\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial x_i}(p, x, u^*)
\]

\[
u^* = \arg\min_u \{ \mathcal{H}(p, x, u) : 0 \leq g(x, u) \}
\]

plus boundary and transversality conditions.
Two Approaches

**Indirect Approach:** Discretize the PMP equations and solve the resulting two point boundary value problem in $2n$ variables.

**Direct Approach:** Discretize the trajectory optimization problem to get a nonlinear program which can be solved by existing software. If we discretize time with step size $h$, then decision variables are $u(0), u(h), u(2h), \ldots, u(T-h)$. Regardless of the state dimension $n$, it requires optimizing a function of $mT/h$ variables subject to constraints. Because of the development of excellent software for solving nonlinear programs, the direct approach has become more popular.
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Discretization of the Optimal Trajectory Problem

\[ \min_{u(0:T)} \sum_{t=0:h:T-1} \bar{l}(x, u) \]

\[ x^+ = \bar{f}(x, u), \quad 0 \leq g(x, u) \]

\[ x(0) = x^0, \quad x(T) = x^T \]

where the discrete dynamics and discrete Lagrangian are defined by Lie differentiation

\[ \bar{f}(x, u) = x + f(x, u)h + L_f(x, u)f(x, u)\frac{h^2}{2} + L^2_f(x, u)f(x, u)\frac{h^3}{6} + \ldots \]

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**Lie differentiation:**

\[ L_f(x, u)h(x, u) = \frac{\partial h}{\partial x}(x, u)f(x, u) \]
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If Lie differentiation is difficult use Runge-Kutta approximations.
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Recall that we have to find the minimum of a function of $mT/h$ variables, $u(0), u(h), u(2h), \ldots, u(T - 1)$. 
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Recall that we have to find the minimum of a function of $mT/h$ variables, $u(0), u(h), u(2h), \ldots, u(T-1)$.

The discretization of the continuous time problem is a form a quadrature so we could use any quadrature rule, e.g., Euler, Trapezoidal, Hermite-Simpson, etc. in either explicit or implicit form.
Efficient Quadrature Rules

Perhaps the most efficient quadrature is Legendre-Gauss (LG). It uses only $N$ nodes to exactly integrate any polynomial of degree $2N - 1$ or less.
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On the standard interval \([-1, 1]\) it takes the form

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\int_{-1}^{1} \phi(t) \, dt = \sum_{i=1}^{N} w_i \phi(t_i)
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where the nodes \( t_i \) are the zeros of the \( N^{th} \) Legendre polynomial \( P_N(t) \).
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But all the nodes $t_i$ are in the open interval $(-1, 1)$ so Legendre-Gauss quadrature is not suitable if there are boundary conditions.
Figure: Legendre Polynomials to Degree 5
Legendre-Gauss-Lobatto (LGL) quadrature is slightly less efficient. It uses $N + 1$ nodes to exactly integrate any polynomial of degree $2N - 1$ or less.
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The weights are \( \frac{2}{n(n-1)} \) at the endpoints and

\[
    w_i = \frac{2}{n(n-1)(P_N(t_i))^2}
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in between.
Gong, Kang and Ross have shown that the pseudospectral method converges for feedback linearizable systems. If $m = 1$ such a system can be transformed to

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\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= f_n(x) + g_n(x)u
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Minimize

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\int_{-1}^{1} l(x(t), u(t)) \, dt + \alpha(x(-1), x(1))
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subject to

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\begin{align*}
0 &= \beta(x(-1), x(1)) \\
0 &\leq \gamma(x(t), u(t))
\end{align*}
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Pseudospectral Trajectory Optimization

Each $x_i(t)$ is approximated by an $N^{th}$ degree interpolating polynomial $\bar{x}_i(t)$. These polynomials are represented by their values at the $N + 1$ LGL nodes,

$$
\bar{x}_i = \begin{bmatrix} \bar{x}_i^0 \\ \vdots \\ \bar{x}_i^N \end{bmatrix}'
$$

$$
\bar{x}_i(t) = \sum_{0}^{N} \bar{x}_i^j \phi_j(t)
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where the $\phi_j(t)$ are the Lagrange polynomials at the LGL nodes.
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where the $\phi_j(t)$ are the Lagrange polynomials at the LGL nodes.

The dynamics is approximated by the equations

$$\bar{x}_{i+1} = D\bar{x}_i, \quad i = 1, \ldots, n - 1$$

$$\bar{u}^j = \frac{(D\bar{x}_n)^j - f(\bar{x}^j)}{g(\bar{x}^j)}$$

so the $N + 1$ decision variables are $\bar{x}_1^0, \ldots, \bar{x}_1^N$. 
Pseudospectral Trajectory Optimization

Multiplication of the interpolated values of a polynomial by the differentiation matrix $D$ yields the interpolating values of its derivative.
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$$u(t) = \frac{\dot{x}_n - f(\bar{x}(t))}{g(\bar{x}(t))}$$

The cost is approximated by a LGL quadrature

$$\sum_{j=0}^{N} l(x^j, u^j) w_j + \alpha(\bar{x}^0, \bar{x}^N)$$

The boundary conditions are approximated by a relaxed version of

$$0 = \beta(\bar{x}^0, \bar{x}^N)$$

and the constraints are approximated by a relaxed version of

$$0 \leq \gamma(\bar{x}^j, \bar{u}^j), \quad j = 0, \ldots, N$$
Pseudospectral Trajectory Optimization and the ISS

SIAM News, Volume 40, Number 7, September 2007
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Pseudospectral Optimal Control Theory Makes Debut Flight, Saves NASA $1M in Under Three Hours

By Wei Kang and Naz Bedrossian
Model Predictive Control

Suppose the problem of minimizing

\[ \int_0^\infty l(x, u) \, dt \]

subject to

\[ \dot{x} = f(x, u) \]
\[ x(0) = x^0 \]
\[ 0 \leq g(x, u) \]

has been discretized into minimizing

\[ \sum_{t=0:h:\infty} \bar{l}(x(t), u(t)) \]

subject to

\[ x^+ = \bar{f}(x, u) \]
\[ x(0) = x^0 \]
\[ 0 \leq g(x, u) \]
Model Predictive Control

Minimization over the infinite horizon is too difficult so choose a time window $T$ and a terminal cost $\pi_T(x)$ defined on a terminal set $\mathcal{X}_T$ which is a compact neighborhood of $x = 0$. 
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Consider the problem of minimizing

$$\sum_{t=0:h:T-h} \bar{l}(x(t), u(t)) + \pi_T(x(T))$$

subject to

$$x^+ = \bar{f}(x, u)$$
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$$0 \leq g(x, u)$$
$$x(T) \in \mathcal{X}_T$$

The decision variables are $u(0), \ldots, u(T - h)$. 
Model Predictive Control

Then pass this nonlinear program to a fast solver to find the optimal $u^0(0), \ldots, u^0(T - h)$. This needs to be done in less than the time step $h$. 

Use the control $u^0(t)$ to get the state to $x_1 = x(h)$. Then between times $h$ and $2h$ solve the problem of minimizing

$$
\sum_{t = h}^{T - h} \bar{L}(x(t), u(t)) + \pi_T(x(T + h))
$$

subject to

$$
x_{t+1} = \bar{f}(x, u(t))
$$

$x(h) = x_1$, $0 \leq g(x, u)$, $x(T + h) \in X_T$ to obtain the optimal $u^1(h), \ldots, u^1(T)$. Use the control $u^1(h)$ to get the state to $x_2 = x(2h)$, etc.
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Use the control \( u^1(h) \) to get the state to \( x^2 = x(2h) \), etc.
The key issues are the following

- If the discrete time system is a discretization of a continuous time system then the time step $h$ must be short enough to accurately approximate it.
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• The horizon \( T \) must be long enough and/or \( X_T \) large enough so that \( x(t + T) \in X_T \).

• The initial guess of \( u^0(0), \ldots, u^0(T - 1) \) that is fed to the solver must be close to optimal else the solver may fail to converge to the true solution.
This is not as much a problem with later initial guesses because we can take $u^0(h), \ldots, u^0(T - h)$ as the initial guess for $u^1(h), \ldots, u^1(T - h)$.  

The ideal terminal cost $\pi_T(x)$ is the optimal cost of the infinite horizon optimal control problem provided that it can be computed on a large enough $X_T$. Then the exact solutions to the finite horizon and infinite horizon optimal control problems are identical.  

If the infinite horizon optimal control law $\kappa_T(x)$ is known on the terminal set $X_T$ then the initial guess for $u^1(T)$ should be $\kappa_T(x_0(T))$ where $\bar{x}_0(T)$ is the $T$th state generated by the last control sequence.
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- Model Predictive Control is a viable alternative to solving Dynamic Programming Equations in low to medium state dimensions for slow processes even when there are state and control constraints.

For a copy of these slides contact ajkrener@nps.edu

Thank you! Questions?
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• It may be possible to use power series methods to compute the terminal cost $\pi_T(x)$ and feedback $\kappa_T(x)$ on a larger terminal set $\mathcal{X}_T$. This may allow us to lengthen the time step $h$ and/or shorten the horizon $T$ so that MPC can be used on faster processes.

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