Computational Issues in Nonlinear Dynamics and Control

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Typical Problems

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- **Numerical Solution of Optimal Control Problems**

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An Important Problem

Infinite Horizon Optimal Control

$$\min_{u(0:\infty)} \int_0^\infty l(x,u)\; dt$$

$$\dot{x} = f(x,u), \qquad x(0) = x^0$$
$$x \in \mathbb{R}^{n\times 1}, \qquad u \in \mathbb{R}^{m\times 1}$$

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Hamiltonian, a function of x, u and a new variable $p \in I\!R^{1\times n}$

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Hamilton Jacobi Bellman Equations

$$\begin{aligned} 0 &= \min_u \mathcal{H}(\frac{\partial \pi}{\partial x}(x), x, u) \\ \kappa(x) &= \operatorname{argmin}_u \mathcal{H}(\frac{\partial \pi}{\partial x}(x), x, u) \end{aligned}$$

Stabilization by Optimization

Problem: Find a feedback $u = \kappa(x)$ so that the closed loop system is (locally) asymptotically stable around $x = 0$.

Solution: Choose a Lagrangian $l(x, u) \geq 0$ and solve the infinite horizon optimal control problem. Under suitable conditions the optimal feedback $u = \kappa(x)$ is stabilizing on some domain around $x = 0$ and this can be verified because the optimal cost $\pi(x) \geq 0$ is a Lyapunov function,

$$\frac{d}{dt}\pi(x(t)) = \frac{\partial \pi}{\partial x}(x(t))f(x(t), \kappa(x(t))) = -l(x(t), \kappa(x(t))) \leq 0$$

Classic Example: LQR

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Theorem: If $Q \geq 0,\ R > 0$, (F,G) stabilizable and $(Q^{1/2},F)$ detectable then there exist a unique nonnegative definite solution P to the Riccati equation and the feedback $u = Kx$ is asymtotically stable, i.e., all the poles of $F + GK$ are in the open left half plane.

Another Important Problem

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$$\min_{u(0:T)} \int_0^T l(t,x,u)\ dt + \pi^T(x(T))$$

$$\begin{aligned} \dot{x} &= f(t,x,u) \\ 0 &= g(x(0), X(T)) \\ u(t) &\in \mathcal{U}(t,x) \end{aligned}$$

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Pontryagin Maximum Principle:

If $x^*(0:T),\ u^*(0:T)$ is optimal then there exists $p:[0,T]\to I\!R^{1\times n}$ such that

$$\begin{aligned} \dot{x}_i^* &= \frac{\partial \mathcal{H}}{\partial p_i}(t,p,x^*,u^*) \\ \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial x_i}(t,p,x^*,u^*) \\ u^* &= \mathrm{argmin}_{u\in\mathcal{U}(T,x^*)} \mathcal{H}(t,p,x^*,u^*) \\ \mathcal{H}(t,p,x,u) &= pf(t,x,u)+l(t,x,u) \end{aligned}$$

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- **first discretize and then analyze.**

Commutative Diagrams?

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\textbf{Infinite Horizon OCP} & \rightarrow & \textbf{Dynamic Program} \\
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\textbf{Finite Horizon OCP} & \rightarrow & \textbf{Nonlinear Program} \\
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\textbf{PMP} & \rightarrow & \textbf{Karush Kuhn Tucker Conditions}
\end{array}$$

Discretization of the HJB equation

For simplicty of exposition assume $n = 2,\ m = 1$. Choose a rectangle around $x = 0$ and partition it with stepsize h . Let $x^{i,j}$ denote the i, j node. Let $\pi^{i,j}$ be the current computed approximation to the optimal cost at the $x^{i,j}$.

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For each i, j solve for the next approximation $\kappa^{i,j}$ to the optimal feedback

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\begin{aligned}
\kappa^{i,j} \;&=\; \operatorname{argmin}_u \Big\{ \left(\pi^{i+1,j} - \pi^{i-1,j}, \pi^{i,j+1} - \pi^{i,j-1}\right) f(x^{i,j}, u) \\
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The boundary condition is $\bar{\pi}^{i_0,j_0} = 0$ where $x^{i_0,j_0} = 0$.

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This is called policy iteration and it is not very efficient because it sweeps through the nodes many times.

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$$x^{i,j} + f(x^{i,j}, u^k)h$$

Then on the state and control grids we have the discrete dynamics

$$\begin{aligned} x^+ &= \bar{f}(x, u) \\ x(0) &= x^{i,j} \end{aligned}$$

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and we minimize by choice of control sequence

$$\begin{aligned} \pi^{i,j} &= \min_{u(0:\infty)} \sum_{t=0:h:\infty} l(x, u)h \\ \kappa^{i,j} &= u^*(0) \end{aligned}$$

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This is again policy iteration but it is still slow.

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Partition the state space into a grid with spacing h and partition time with spacing k . Construct a Controlled Markov Chain with transition probability $p(x^1|x^0,u)$ from gridpoint x^0 to grid point x^1 with control u. Choose a search radius r and define

$$p(x^1|x^0,u) = \frac{\exp(-(\|x^1 - x^0 - (f(x^0,u)k\|^2)}{\rho(x^0,u)}$$

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The cost is defined to be the expected value of

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In dimensions $n = 2, 3$ and $m = 1, 2$ this is a feasible method.

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But in higher dimensions it difficult to implement.

Eikonal Equation

Suppose that the speed of propagation $c(x) > 0$ through a medium varies with location. Consider any path $x(t)$ between the source $x^0 = 0$ and x^1. Then the propagation time along this path is $\int_0^t 1\ d\tau$ so the Lagrangian $l(x,u) = 1$.

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which reduce to

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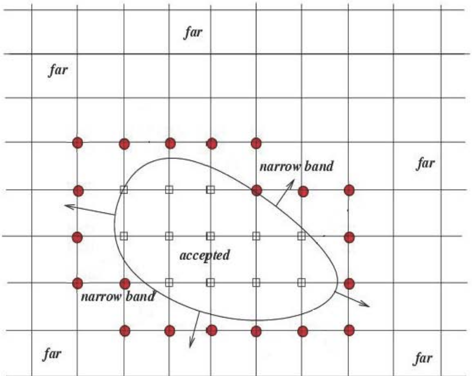
Partitition the nodes into three families called accepted, narrow band and far. Initially the only node in accepted family is the origin and $\pi^{i_0,j_0} = 0$.



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Assume that $\pi^{i,j}$ has been computed for all the accepted nodes.

For each node $x^{i,j}$ in the narrow band compute the rectilinear path to a node $x^{r,s}$ in the accepted region that minimizes the sum of travel time along the path plus $\pi^{r,s}$.

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Different implimentations of FMM use different ways of computing the sum of travel time along the path plus $\pi^{r,s}$.

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For each node $x^{i,j}$ in the narrow band compute the rectilinear path to a node $x^{r,s}$ in the accepted region that minimizes the sum of travel time along the path plus $\pi^{r,s}$.

This is typically done using Dykstra's method for finding the shortest path on a graph.

Then accept the narrow band node that minimizes this sum.

Repeat until the accepted nodes cover the region where the solution is desired.

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The FFM has been generalized to other optimal control problems but computing the minimum sums is more complicated because not every rectilinear path is feasible.

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Practical optimal control problems usually have state dimension larger than 2 or 3. For example, the attitude control problem for a spacecraft has state dimension $n = 6$ and control dimension typically $m = 3$. The position and attitude control problem for an airplane has state dimension $n = 12$ and control dimension at least $m = 4$.

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Consider trying to apply a grid based method.. For the solution to be reasonably accurate we would need a substantial number of grid points in each coordinate direction, e.g., 10^2. Then the total number of grid points is 10^{12} for attitude control and 10^{24} for position and attitude control. If we can process 100 nodes a second that works out to about 300 years for attitude control and $3 \cdot 10^{14}$ years for position and attitude control.

HJB Equations and Conservation Laws

Suppose we have a time varying problem of the form

$$\begin{aligned}\dot{x} &= f(t,x) + g(t,x)u \\ l(t,x,u) &= q(t,x) + \frac{1}{2}u'R(t,x)u\end{aligned}$$

Then the HJB PDEs reduce to the HJ PDE

$$0 = \frac{\partial \pi}{\partial t} + \frac{\partial \pi}{\partial x}f - \frac{1}{2}\frac{\partial \pi}{\partial x}gRg'\left(\frac{\partial \pi}{\partial x}\right)' + q$$

Let $p = \frac{\partial \pi}{\partial x}$ and take the Jacobian of the HJ equation to obtain the conservation law

$$0 = \frac{\partial p}{\partial t} + \frac{\partial}{\partial x}F(t,x,p)$$

where the flux term is

$$F(t,x,p) = pf - \frac{1}{2}pgRg'p' + q$$

HJB Equations and Conservation Laws

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Another is that we are looking for a solution of the conservation law that is a closed one form,

$$\frac{\partial p_i}{\partial x_j} = \frac{\partial p_j}{\partial x_i}$$

Invariant Manifold Methods

Hamilton Differential Equations

$$\dot{x}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial x_i}$$

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So we can compute $\pi(x)$ by computing the stable manifold of the Hamiltonian dynamics.

Hauser-Osinga Method

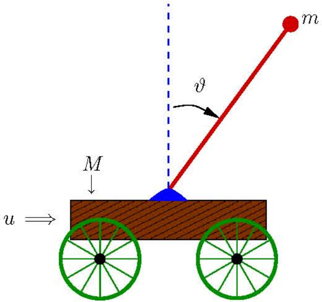

Figure 1: *Sketch of the balanced planar pendulum on a moving cart.*

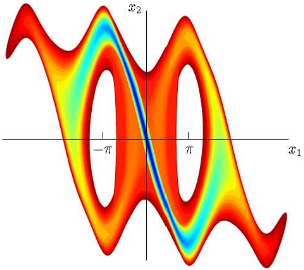

Figure 2: *Each point is colored according to how high the cost of getting to the origin using this point as initial condition. The cost increases as the color changes from blue to red.*

Min-Plus Methods

The min-plus semiring is defined as follows

$$\begin{aligned} \boldsymbol{\xi} \oplus \boldsymbol{\zeta} &= \mathbf{min}\{\boldsymbol{\xi}, \boldsymbol{\zeta}\} \\ \boldsymbol{\xi} \otimes \boldsymbol{\zeta} &= \boldsymbol{\xi} + \boldsymbol{\zeta} \end{aligned}$$

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But is linear in the min-plus sense and $\pi(x)$ is an eigenvector corresponding to the eigenvalue $\mathbf{0}$ which is the $\otimes$ identity.

$$0 \otimes \pi(x) = \mathcal{S}_T(\pi(x)$$

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To make the calculation finite dimensional $\pi(x)$ is chosen as a min-plus combination of basis functions.

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The number of basis functions needed for a given accuracy is exponential in the state dimension n but it is probably grows slower than the number of grid points.

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But the computational cost of the method grows exponentialy in the number of LQR Hamiltonians. The number of LQR Hamiltonians needed probably increases with the state dimension but more slowy than the number of gridpoints does.

It also suffers from a curse of complexity as it requires computing the pointwise maxima (or minima) of a large number of functions which can be expensive.

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If $h_1 = 0.01$ then $h_3 \approx 0.1$ so the number of grid points that is needed for a given accuracy is reduced by a factor of 10 in each dimension. If the state dimension is n then the reduction in grid points is by a factor of 10^n .

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If the third order method takes $k_3(n)$ times longer to compute for each node then the reduction in computational time is by the factor $\frac{10^n}{k_3(n)}$. Typically $k_3(n)$ is polynomial in n.

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There are diminishing returns as we go to higher orders. Consider a fifth order method with step size h_5 . Then for same level of accuracy

$$h_1^2 \approx h_5^6 \quad \text{so} \quad h_5 \approx h_1^{1/3}$$
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Suppose we have a first order method $M_1(h)$ for solving a problem using stepsize h . If the problem and the method are smooth enough then we expect that the error is a power series in h with lowest order term a constant times h^2 . Let α denote the true solution then with steps sizes h and $2h$

$$\begin{aligned}
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Szpiro and Dupuis have applied this technique to HJB equations.

Singular PDEs

A first order quasilinear PDE

$$0 = \frac{\partial \phi}{\partial x}(x)a(x) + b(x, \phi(x))$$

is said to be singular at $x = 0$ if $a(0) = 0$. Usually $b(0,0) = 0$.

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We expand in power series.

$$\begin{aligned} a(x) &= Ax + a^{[2]}(x) + a^{[3]}(x) + \ldots \\ b(x,\phi(x)) &= Cx + B\phi(x) + (b(x,\phi(x)))^{[2]} + (b(x,\phi(x)))^{[3]} + \ldots \\ \phi(x) &= Tx + \phi^{[2]}(x) + \phi^{[3]}(x) + \ldots \end{aligned}$$

where $(\cdot)^{[d]}$ denotes terms homogeneous of degree d.

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The higher degrees terms can be found in a similar fashion.

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Singular PDEs

Many of the most important PDEs of nonlinear dynamics and control are essentially singular first order quasilinear including

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Discrete time and time varying problems can also be solved by similar power series methods.

In discrete time the degree two nonresonance conditions are

$$\alpha_{i_1}\alpha_{i_2} \neq \beta_j$$

Al'brecht's Method

Al'brecht developed the power series method for HJB equations for the optimal cost and optimal feedback,

$$\begin{aligned}\pi(x) &= \frac{1}{2}x'Px + \pi^{[3]}(x) + \pi^{[4]}(x) + \dots \\ \kappa(x) &= Kx + \kappa^{[2]}(x) + \kappa^{[3]}(x) + \dots\end{aligned}$$

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This guarantees there are no resonances so the higher degree terms of π, κ can be found by solving invertible linear equations.

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This method has been implimented in the Nonlinear Systems Toolbox.

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The HJB equations can be solved to degree 4 in $\pi(x)$ and degree 3 in $\kappa(x)$ for systems with state dimension $n = 25$ and control dimension $m = 8$ on a lap top.

Pros and Cons of Power Series Methods

- **Power Series Methods can also be used for discrete time and time varing problems.**

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Patchy Methods

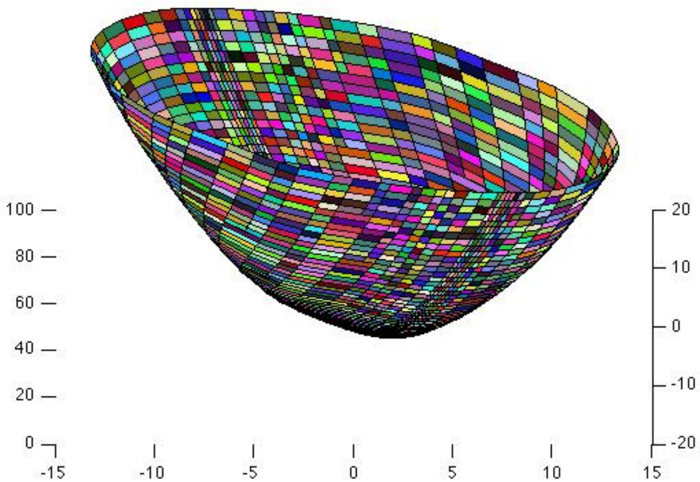
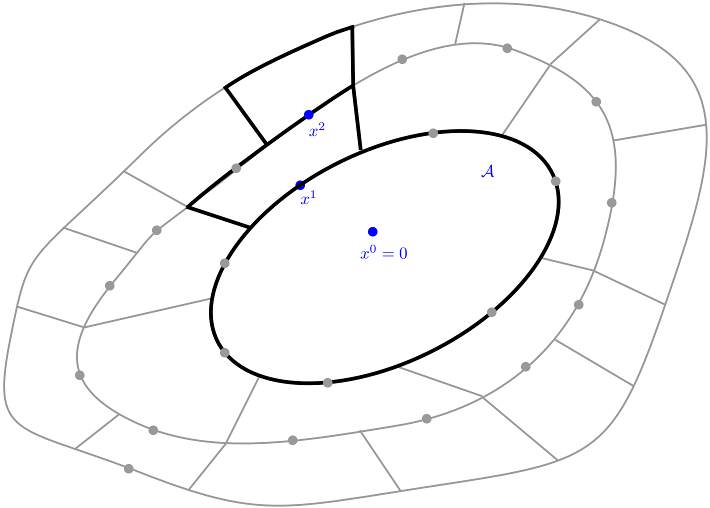


Figure : Optimal Cost of Inverting a Pendulum by a Torque at its Axis

Sequence of Patches



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Patch Calculation

The HJB equations are not singular away from the origin. The map

$$\pi^{[d+1]} \mapsto \frac{\partial \pi^{[d+1]}}{\partial x}(x) f(x^1, u^1)$$

takes a polynomial of degree $d+1$ to a polynomial of degree d.

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Under suitable assumptions there is one positive root and one negative root. We take the positive root.

Invert a Pendulum

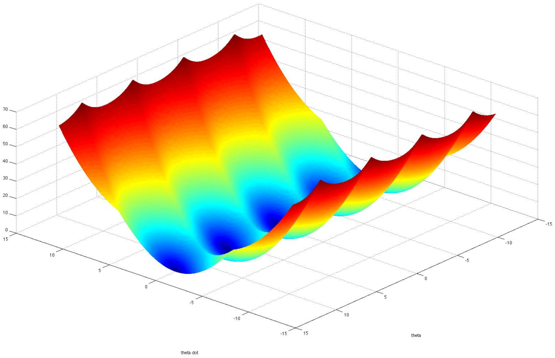

Figure : Periodicity of the Optimal Cost

The left axis is $-15 \leq \dot{\theta} \leq 15$ and the right axis is $-15 \leq \theta \leq 15$. From points on the ridges there are two optimal trajectories, one going to the left well and the other going to the right well.

Adaptive Algorithm

The algorithm is adaptive. It splits a patch in two when the relative residue of the first HJB equation is too high at the lower corners of a patch. It also lowers the upper level of a ring of patches if the relative residue is too high on it.

Ring	1	2	3	4
Initial Patch Level	0.64	1.21	1.96	2.89
Final Patch Level	0.36	0.63	1.38	2.23
Initial No. Patches	1	24	26	26
Final No. Patches	1	26	26	28

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Only the first ten patch levels were adjusted down.

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The last ring (34) contains 78 patches.

Patchy Hauser Osinga Pendulum

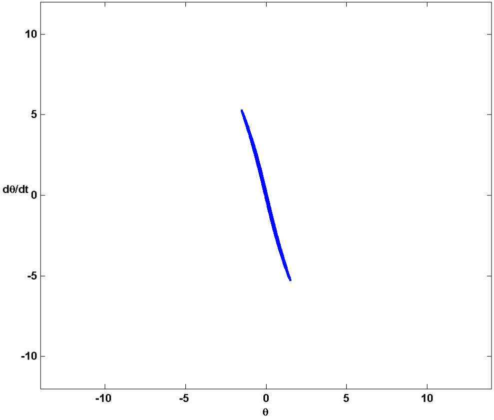

Figure : Patchy Optimal Cost to Level Set 70

Patchy Hauser Osinga Pendulum

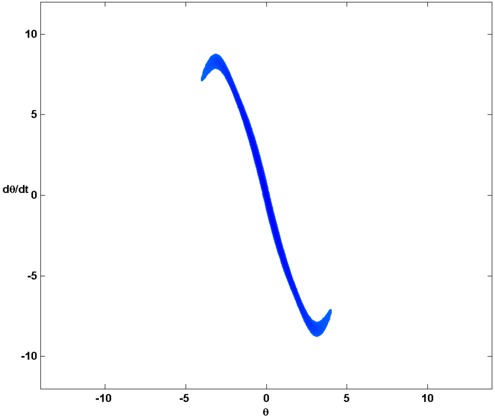

Figure : Patchy Optimal Cost to Level Set 140

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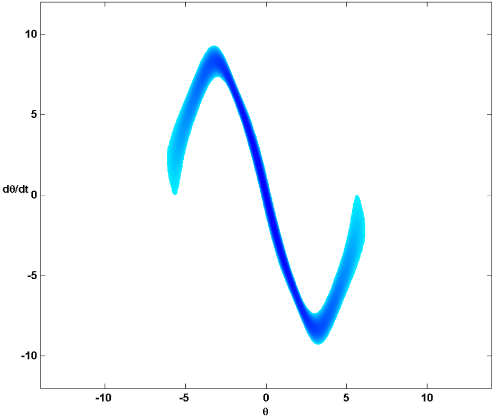

Figure : Patchy Optimal Cost to Level Set 210

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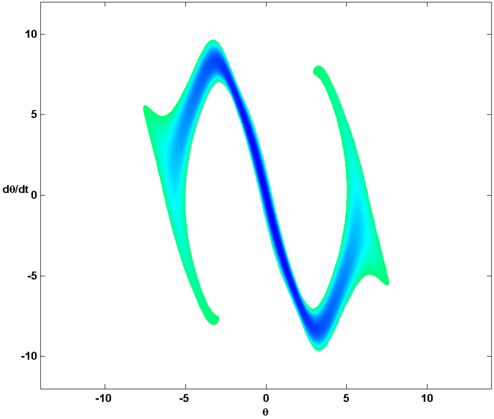

Figure : Patchy Optimal Cost to Level Set 280

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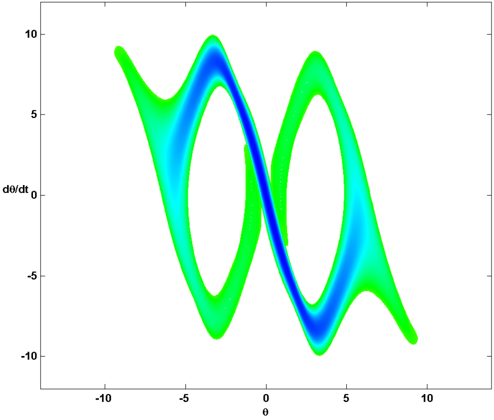

Figure : Patchy Optimal Cost to Level Set 350

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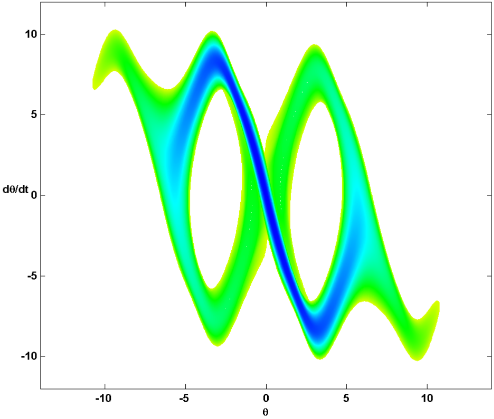

Figure : Patchy Optimal Cost to Level Set 420

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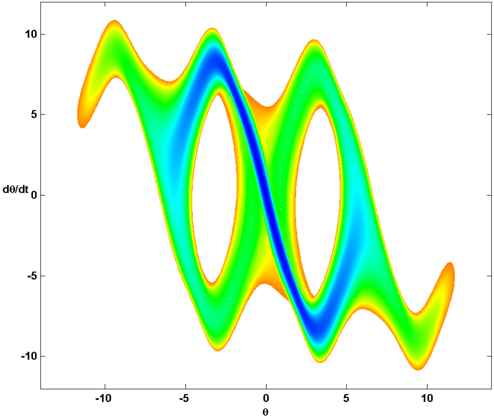

Figure : Patchy Optimal Cost to Level Set 490

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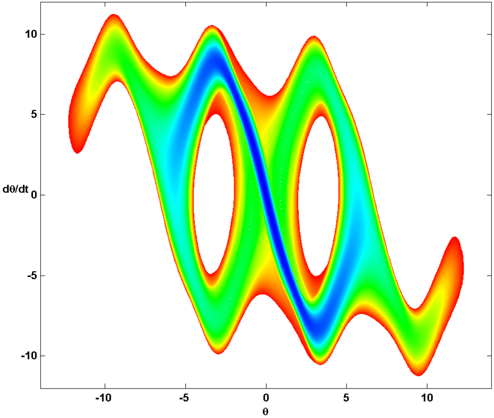

Figure : Patchy Optimal Cost to Level Set 566

Error Comparison

A nonlinear change of state coordinates on an LQR problem yields a nonlinear optimal control problem.

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Here are the errors between the true optimal cost and the computed optimal cost which is of degree $d+1$.

	Max Error	Max Rel Error	Error Factor
LQR ($d=1$)	0.3543	0.8860	54.56
Al'brecht ($d=3$)	0.1636	0.4101	25.16
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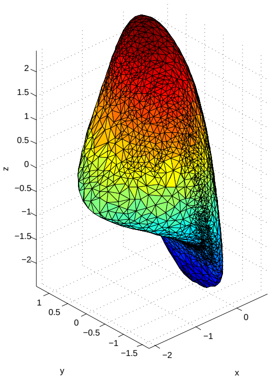
This shows that the patchy method can be very accurate and it is parallelizable.

Three Dimensional Example

Here is a level set of the patchy method applied to a three dimensional problem



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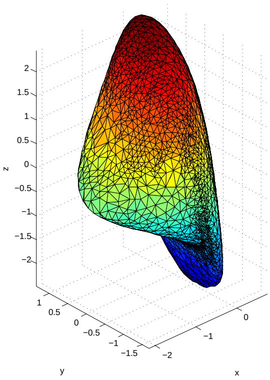

Figure : Level Set 55

The complexity of keeping track of the patches probably makes the patchy method infeasible in higher dimesions.

Conclusions

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Similar statements are true for the other PDEs of nonlinear control.

Trajectory Optimization

Typical Problem

$$\min_{u(0:T)} \int_0^T l(x,u)\ dt$$

subject to

$$\begin{aligned} \dot{x} &= f(x,u), &\quad 0 &\le g(x,u) \\ x(0) &= x^0, &\quad x(T) &= x^T \end{aligned}$$

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Pontryagin Minimum Principle: There exists $p(0:T) \in I\!R^{1\times n}$

$$\begin{aligned} \dot{x}_i &= \frac{\partial \mathcal{H}}{\partial p_i}(p,x,u^*) \\ \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial x_i}(p,x,u^*) \\ u^* &= \operatorname{argmin}_u \{\mathcal{H}(p,x,u) : 0 \le g(x,u)\} \end{aligned}$$

plus boundary and transversality conditions.

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Indirect Appoach: **Discretize the PMP equations and solve the resulting two point boundary value problem in $2n$ variables.**

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Because of the development of excellent software for solving nonlinear programs, the direct approach has become more popular.

Discretization of the Optimal Trajectory Problem

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\begin{array}{rcl}
& \min\limits_{u(0:T)} \quad \sum_{t=0:h:T-1} & \bar{l}(x,u) \\
x^+ = \bar{f}(x,u), & & 0 \le g(x,u) \\
x(0) = x^0, & & x(T) = x^T
\end{array}
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where the discrete dynamics and discrete Lagrangian are defined by Lie differentiation

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Lie differentiation: $L_{f(x,u)} h(x,u) = \dfrac{\partial h}{\partial x}(x,u) f(x,u)$

The Euler approximation stops at the h terms.

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\bar{f}(x,u) &= x + f(x,u)h + L_{f(x,u)} f(x,u)\frac{h^2}{2} + L^2_{f(x,u)} f(x,u)\frac{h^3}{6} + \\
\bar{l}(x,u) &= l(x,u)h + L_{f(x,u)} l(x,u)\frac{h^2}{2} + L^2_{f(x,u)} l(x,u)\frac{h^3}{6} + \dots
\end{aligned}
$$

Lie differentiation: $L_{f(x,u)}h(x,u) = \frac{\partial h}{\partial x}(x,u)f(x,u)$

The Euler approximation stops at the h terms.

If Lie differentiation is difficult use Runge-Kutta approximations.

Discretization of the Optimal Trajectory Problem

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Recall that we have to find the minimum of a function of mT/h variables, $u(0), u(h), u(2h), \ldots, u(T-1)$.

The discretization of the continuous time problem is a form a quadrature so we could use any quadrature rule, e.g., Euler, Trapezoidal, Hermite-Simpson, etc. in either explicit or implicit form.

Efficient Quadrature Rules

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But all the nodes t_i are in the open interval $(-1,1)$ so Legendre-Gauss quadrature is not suitable if there are boundary conditions.

Efficient Quadrature Rules

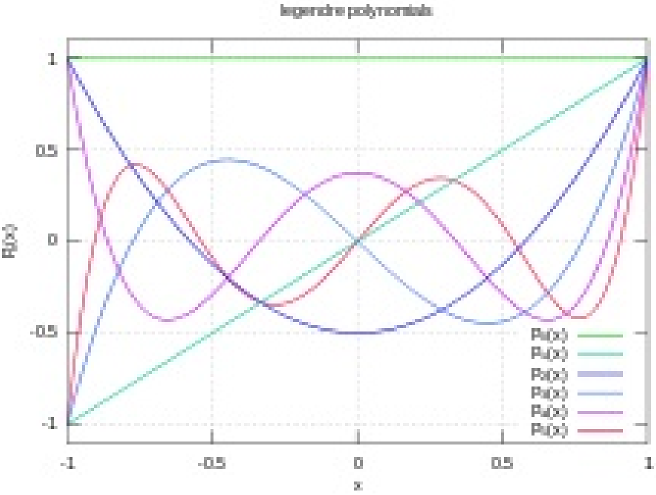

Figure : Legendre Polynomials to Degree 5

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The weights are $\frac{2}{n(n-1)}$ at the endpoints and

$$w_i = \frac{2}{n(n-1)(P_N(t_i))^2}$$

in between.

Pseudospectral Trajectory Optimization

Gong, Kang and Ross have shown that the pseudospectral method converges for feedback linearizable systems. If $m = 1$ such a system can be transformed to

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
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Minimize

$$\int_{-1}^{1} l(x(t), u(t))\, dt + \alpha(x(-1), x(1))$$

subject to

$$\begin{aligned}
0 &= \beta(x(-1), x(1)) \\
0 &\leq \gamma(x(t), u(t))
\end{aligned}$$

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Each $x_i(t)$ is approximated by an N^{th} degree interpolating polynomial $\bar{x}_i(t)$. These polynomials are represented by their values at the $N+1$ LGL nodes,

$$\begin{aligned} \bar{x}_i &= \left[\begin{array}{ccc} \bar{x}_i^0 & \dots & \bar{x}_i^N \end{array} \right]' \\ \bar{x}_i(t) &= \sum_0^N \bar{x}_i^j \phi_j(t) \end{aligned}$$

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where the $\phi_j(t)$ are the Lagrange polynomials at the LGL nodes.

The dynamics is approximated by the equations

$$\begin{aligned} \bar{x}_{i+1} &= D\bar{x}_i, \qquad i = 1, \ldots, n-1 \\ \bar{u}^j &= \frac{(D\bar{x}_n)^j - f(\bar{x}^j)}{g(\bar{x}^j)} \end{aligned}$$

so the $N+1$ decision variables are $\bar{x}_1^0, \ldots, \bar{x}_1^N$.

Multiplication of the interpolated values of a polynomial by the differentiation matrix D yields the interpolating values of its derivative.

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$$u(t) = \frac{\dot{\bar{x}}_n - f(\bar{x}(t))}{g(\bar{x}(t))}$$

The cost is approximated by a LGL quadrature

$$\sum_{j=0}^{N} l(x^j, u^j) w_j + \alpha(\bar{x}^0, \bar{x}^N)$$

The boundary conditions are approximated by a relaxed version of

$$0 = \beta(\bar{x}^0, \bar{x}^N)$$

and the constraints are approximated by a relaxed version of

$$0 \le \gamma(\bar{x}^j, \bar{u}^j), \qquad j = 0, \dots, N$$

Pseudospectral Trajectory Optimization and the ISS

SIAM News, Volume 40, Number 7, September 2007

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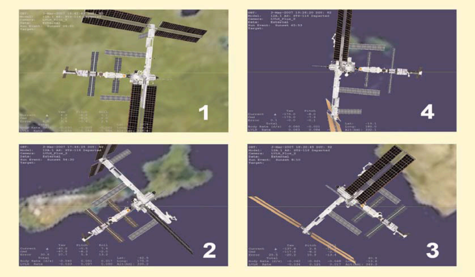
Pseudospectral Optimal Control Theory Makes Debut Flight, Saves NASA $1M in Under Three Hours

By Wei Kang and Naz Bedrossian



Model Predictive Control

Suppose the problem of minimizing

$$\int_0^\infty l(x,u)\ dt$$

subject to

$$\begin{aligned} \dot{x} &= f(x,u) \\ x(0) &= x^0 \\ 0 &\le g(x,u) \end{aligned}$$

has been discretized into minimizing

$$\sum_{t=0:h:\infty} \bar{l}(x(t),u(t))$$

subject to

$$\begin{aligned} x^+ &= \bar{f}(x,u) \\ x(0) &= x^0 \\ 0 &\le g(x,u) \end{aligned}$$

Model Predictive Control

Minimization over the infinite horizon is too difficult so choose a time window T and a terminal cost $\pi_T(x)$ defined on a terminal set $\mathcal{X}_T$ which is a compact neighborhood of $x = 0$.

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The decision variables are $u(0), \ldots, u(T-h)$.

Model Predictive Control

Then pass this nonlinear program to a fast solver to find the optimal $u^0(0), \dots, u^0(T-h)$. This needs to be done in less than the time step h.

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to obtain the optimal $u^1(h), \ldots, u^1(T)$.

Use the control $u^1(h)$ to get the state to $x^2 = x(2h)$, etc.

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The key issues are the following

- **If the discrete time system is a discretization of a continuous time system then the time step h must be short enough to accurately approximate it.**

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- **The initial guess of $u^0(0), \ldots, u^0(T-1)$ that is fed to the solver must be close to optimal else the solver may fail to converge to the true solution.**

Model Predictive Control

- **This is not as much a problem with later initial guesses because we can take $u^0(h), \ldots, u^0(T-h)$ as the initial guess for $u^1(h), \ldots, u^1(T-h)$.**

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- **If the infinite horizon optimal control law $\kappa_T(x)$ is known on the terminal set $\mathcal{X}_T$ then the initial guess for $u^1(T)$ should be $\kappa_T(x^0(T))$ where $\bar{x}^0(T)$ is the T^{th} state generated by the last control sequence. $u^0(0), \ldots, u^0(T-1)$**

Concluding Remarks

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- **Thank you! Questions?**

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