Model Predictive Regulation

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Research supported by AFOSR

Stabilization around an Operating Point

Controlled Dynamics in Discrete Time

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where $x,\ u$ are $n,\ m$ dimensional.

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0 \leq g(x, u)
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We seek a feedback law

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u = \kappa(x)
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to stabilize the system around the operating point.

Optimal Control Problem

A standard approach is to recast this as an infinite horizon optimal control problem

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The running cost may be given by economic considerations or just chosen so that $x(t) \to 0$ without using too much $u(t)$, e.g.

$$l(x, u) = x'Qx + u'Ru$$

Dynamic Programming Equation

$\pi(x)$, **Optimal Cost given** $x(0) = x$
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Dynamic Programming (DP) Equations

$$\begin{aligned}\pi(x) &= \min_u \{\pi(f(x,u)) + l(x,u)\} \\ \kappa(x) &= \operatorname{argmin}_u \{\pi(f(x,u)) + l(x,u)\}\end{aligned}$$

where the minimum is over all admissible controls

$$\{u : 0 \leq g(f(x,u),u)\}$$

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These equations are notoriously difficult to solve if the state dimension n is greater than 2 .

Stability

The optimal cost $\pi(x)$ is a Lyapunov function which ensures the stability of the closed loop system

$$\pi(x^+(t)) \leq \pi(x(t))$$

provided

$$\begin{aligned} \pi(x) &> 0 \quad \text{if } x \neq 0 \\ l(x,u) &\geq 0 \quad \text{if } x \neq 0 \end{aligned}$$

and other conditions are satisfied.

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subject to

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The terminal cost $\pi^T(x)$ may only be defined in some compact neighborhood $\mathcal{X}$ of $x^0 = 0$ so an extra constraint is needed,

$$x(t+T) \in \mathcal{X}$$

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- **The terminal cost must be a control Lyapunov function for the dynamics.**
- **The ideal terminal cost is the optimal cost of the infinite horizon optimal control problem provided that it can be computed on a large enough $\mathcal{X}$. Then the solutions to the finite horizon and infinite horizon optimal control problems are identical.**

Regulation

In the regulation problem we are given a plant

$$\begin{aligned} x^+ &= f(x,u,w) \\ y &= h(x,u,w) \end{aligned}$$

that it is affected by an external signal $w(t)$ that might be a command or a disturbance. The dimension of y is p and we usually assume that $p = m$.

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The goal is to find a feedforward and feedback $u = \kappa(x, w)$ such that $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

Exosystem

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Nonlinear Exosystem:

$$w^+ \;=\; a(w)$$

The dimension of w is k .

A usual assumption is that the exosytem is neutrally stable in some sense, e.g., all the eigenvalues of

$$\frac{\partial a}{\partial w}(0)$$

are on the unit circle.

Francis Byrnes Isidori Equation

The first step in nonlinear regulation is to solve the discrete time Francis Byrnes Isidori (FBI) equations.
Find $x = \phi(w)$ and $u = \alpha(w)$ such that

$$\begin{aligned} f(\phi(w), \alpha(w), w) &= \phi(a(w)) \\ h(\phi(w), \alpha(w), w) &= 0 \end{aligned}$$

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Then the graph of $x = \phi(w)$ is a controlled invariant submanifold of (x, w) space on which $y = 0$.

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Choose a suitable running cost $l(z, v)$ **and**

$$\min \sum_{t=0}^{\infty} l(z(t), v(t))$$

subject to

$$\begin{aligned} z^+ &= \bar{f}(z, v, w) = f(\phi(w) + z, \alpha(w) + v, w) - \phi(a(w)) \\ w^+ &= a(w) \end{aligned}$$

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Despite this Krener showed in continuous time under suitable conditions that a nice solution exists locally around $(x^0, w^0) = (0, 0)$. The optimal cost $\rho(z, w)$ is a Lyapunov function for the z dynamics under the feedforward and feedback control law $v = \beta(z, w)$.

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Hence $y(t) \to 0$ under the combined control law

$$u \;=\; \kappa(x, w) = \alpha(w) + \beta(x - \phi(w), w)$$

and the optimal cost is

$$\pi(x, w) \;=\; \rho(x - \phi(w), w)$$

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In particular the running cost $l(z, v)$ should be nonnegative definite in z and positive definite in v . One possibility is

$$l(z, v) = z' Q z + v' R v$$

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The key thing is to choose a running cost $l(x(t), u(t), w(t))$ that is zero when $y(t) = 0$. It should also be nonegative definite in $z(t)$ and positive definite in $v(t)$ even though we might not know what $v = u - \alpha(w)$ is.

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How do we do this?

By making l a function of $y(t), y(t+1), \ldots, y(t+r)$ where r is the relative degree of the plant.

Relative Degree

For simplicity of exposition we assume a SISO system, $m = p = 1$.

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Define a family of functions $h^{(j)}(x, u, w)$ as

$$\begin{aligned} h^{(0)}(x, u, w) &= h(x, u, w) \\ h^{(j)}(x, u, w) &= h^{(j-1)}(f(x, u, w), u, a(w)) \end{aligned}$$

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The plant and exosystem

$$\begin{aligned} x^+ &= f(x, u, w) \\ w^+ &= a(w) \\ y &= h(x, u, w) \end{aligned}$$

have well-defined relative degree r if for all x, u, w

$$\begin{aligned} \frac{\partial h^{(j)}}{\partial u}(x, u, w) \quad &= 0 \quad \text{if} \quad 0 \leq j < r \\ &\neq 0 \quad \text{if} \quad j = r \end{aligned}$$

In other words $y(t + r)$ is the first output influenced by $u(t)$.

Running Cost

Then we can choose the running cost as

$$l(x,u,w) \;=\; \sum_{j=0}^{r} \gamma_j (h^{(j)}(x,u,w))^2$$

where $\gamma_j \geq 0$ **and in particular** $\gamma_0 > 0,\ \gamma_r > 0$ **. Or a similar** l **.**

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Clearly if exact regulation is achieved at time t , i.e.,

$$0 \;=\; y(t) = y(t+1) = y(t+2) = \cdots$$

then the running cost is zero from t on.

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This is to difficult to solve via the DP equations.

Finite Horizon Optimal Regulation

Instead if $x(t) = x^t,\ w(t) = w^t$ then we consider the finite horizon optimal control problem

$$\min \sum_{s=t}^{t+T-1} l(x(t), u(t), w(t)) + \pi^T(x(t+T), w(t+T)$$

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where the terminal cost $\pi^T(x, w)$ is defined on $\mathcal{X} \times \mathcal{W}$.

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Falugi and Mayne (CDC13) have proposed a two step method for tracking a periodic signal.

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- The set $\mathcal{W}$ must be invariant under the exosystem dynamics and large enough to contain all possible $w(t)$.
- The terminal cost must be a control Lyapunov function for the closed loop dynamics. If $(x, w) \in \mathcal{X} \times \mathcal{W}$ such that $h(x, u, w) > 0$ for all admissible u then

$$\begin{aligned} \pi(x, w) &> 0 \\ \pi(x, w) &\geq \min_u \pi(f(x, u, w), a(w)) \end{aligned}$$

Ideal Terminal Cost

- **The ideal terminal cost is the optimal cost $\pi(x, w)$ of the infinite horizon optimal control problem provided that it can be computed on a large enough $\mathcal{X} \times \mathcal{W}$. Then the solutions to the finite horizon and infinite horizon optimal regulation problems are identical.**

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- **Suppose the FBI equations are solvable for $\phi(w),\ \alpha(w)$ for $w \in \mathcal{W}$ and the HJB equations for the transverse optimal control problem are solvable for $\rho(z,w),\ \beta(z,w)$ for $z \in (\mathcal{X} - \phi(\mathcal{W}))\,,\ w \in \mathcal{W}$ then the ideal terminal cost is**

$$\pi^T(x,w) \;=\; \rho(x-\phi(w),w)$$

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$$\pi^T(x, w) \ = \ \rho(x - \phi(w), w)$$

Later we shall present a method for calculating these functions on a reasonably large $\mathcal{X} \times \mathcal{W}$.

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We return to the infinite horizon optimal regulation problem

$$\min \sum_{t=0}^{\infty} l(x(t), u(t), w(t))$$

subject to

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Suppose that the DP equations are solvable for the optimal cost $\pi(x, w)$ and optimal feedback $u = \kappa(x, w)$.

Generalized FBI Equations

A subset of (x, w) space is said to satisfy the generalized FBI Equations (gFBI) if there exists a feedforward and feedack $u = \kappa(x, w)$ such that the subset is forward invariant under the close loop dynamics

$$\begin{aligned} x^+ &= f(x, \kappa(x, w), w) \\ w^+ &= a(w) \end{aligned}$$

and $y = 0$ on this subset

$$0 = h(x, \kappa(x, w), w)$$

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and $y = 0$ on this subset

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0 = h(x, \kappa(x, w), w)
$$

The zero set of $\pi(x, w)$,

$$
\mathcal{Z} = \{(x, w) : \pi(x, w) = 0\}
$$

is obviously a solution of the gFBI equations.

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Suppose that the FBI equations are solvable for
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What is the relationship of $\{(x, w) : x = \phi(w)\}$ **to** $\mathcal{Z}$ **?**

Simple Linear Example

Plant, $n = 3,\ m = 1,\ p = 1$

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\begin{aligned}
x^+ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix} u \\
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Two exosystem poles at $\pm i$ **.**
There are no plant zero, exosystem pole resonances so the Francis equations are solvable.

Simple Linear Example

Suppose we let

$$l(x(t),u(t),w(t)) \;=\; (y(t))^2 + (y^{(2)}(t))^2 = (y(t))^2 + (y(t+2))^2$$

Then the solution to the DP equations for the infinite horizon optimal control problem is

$$\pi(x,w) \;=\; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ w_1 \\ w_2 \end{bmatrix}' \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ w_1 \\ w_2 \end{bmatrix}$$

$$\kappa(x,w) \;=\; -\begin{bmatrix} 0 & 0 & 1 & 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ w_1 \\ w_2 \end{bmatrix}$$

Simple Linear Example

The zero set $\mathcal{Z}$ of $\pi(x, w)$ is a three dimensional subspace of (x, w) space given by the equations

$$\begin{aligned} 0 &= x_1 - w_1 \\ 0 &= x_2 + w_2 \end{aligned}$$

In the terminology of Wonham this is $\mathcal{V}^*$, the maximal $A,\ B$ invariant subspace in the kernel of C for combined $x,\ w$ system.

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The graph of the solution to the Francis equations is a two dimensional subspace of (x,w) space given by the equations

$$\begin{aligned} 0 &= x_1 - w_1 \\ 0 &= x_2 + w_2 \\ 0 &= x_3 + 0.2w_1 + 0.4w_2 \end{aligned}$$

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In this example the spectrum of the optimal closed loop dynamics on $\mathcal{Z}$ is $0,0,-0.5$. The Francis dynamics on the graph of $x=\phi(w)$ has spectrum $0,0$ so optimal trajectories in $\mathcal{Z}$ converge to the graph of $x=\phi(w)$ according to the extra eigenvalue -0.5 which is the zero of the plant. In other words at each time step the magnitude of quantity $x_3+0.2w_1+0.4w_2$ is halved and its sign is flipped.

Questions

Given a nonlinear plant and nonlinear exosystem

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- **Can we use MPC techniques to compute the solution to the infinite horizon optimal control problem?**
- **How do we choose the terminal sets $\mathcal{X}$, $\mathcal{W}$ and the terminal cost $\pi^T(x,w)$?**

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As we mentioned above the ideal choice for the terminal cost $\pi^T(x, w)$ is the solution $\pi(x, w)$ to the infinite horizon optimal control problem. But this hard to compute for several reasons.

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- **The infinite horizon optimal control problem is nonstandard because we have no control over w . Standard software, e.g., Matlab's dare.m, cannot solve the resulting algebraic Riccati equation even in the linear quadratic case.**

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The desired solution is

$$\begin{aligned}
\pi(x, w) &= \rho(x - \phi(w), w) \\
\kappa(x, w) &= \alpha(w) + \beta(x - \phi(w), w)
\end{aligned}$$

Approximate Solution of the FBI Equations by Taylor Polynomials

Huang and Rugh showed that the Taylor polynomials of the solution to the FBI equations can be solved around the operating point $(x^0, w^0) = (0, 0)$ by solving a sequence of linear algebraic equations for the coefficients.

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Approximate Solution of the Transverse DP Equations by Taylor Polynomials

Using Al'brecht's method we can find the Taylor polynomials of the solutions to the transverse DP equations around the operating point $(x^0, w^0) = (0, 0)$. At the lowest level this requires solving a discrete time linear quadratic regulator. Then we solve a sequence of linear algebraic equations for the higher coefficients.

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We compute the set $\mathcal{X} \times \mathcal{W}$ on which this $\pi^T(x, w)$ is a control Lyapunov function and use MPR to solve the infinite horizon optimal control problem.

Example 2

Plant:

$$\begin{bmatrix} x_1^+ \\ x_2^+ \\ x_3^+ \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix} u + \begin{bmatrix} 0 \\ -2x_1^3 + x_2^3 - \sin(x_1) \\ -x_1^2 - \sin(x_1) \end{bmatrix}$$

Exosystem:

$$\begin{bmatrix} w_1^+ \\ w_2^+ \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Output: $a^2 + b^2 = 1$

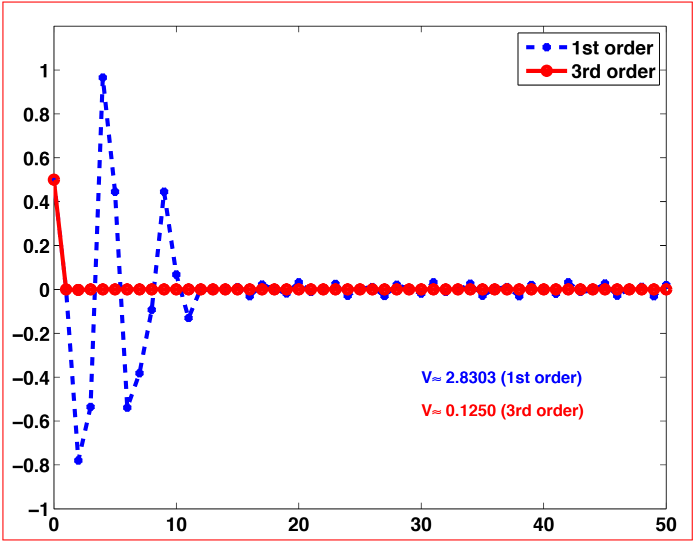
$$y = x_1 - w_1$$

Relative degree $r = 2$.
Running Cost:

$$l(x, u, w) = (h(x_1, w_1))^2 + (h_2^{(2)}(x, u, w))^2$$

Example 2

$$y(t) = x_1(t) - w_1(t),\ 0 \le t \le 50$$



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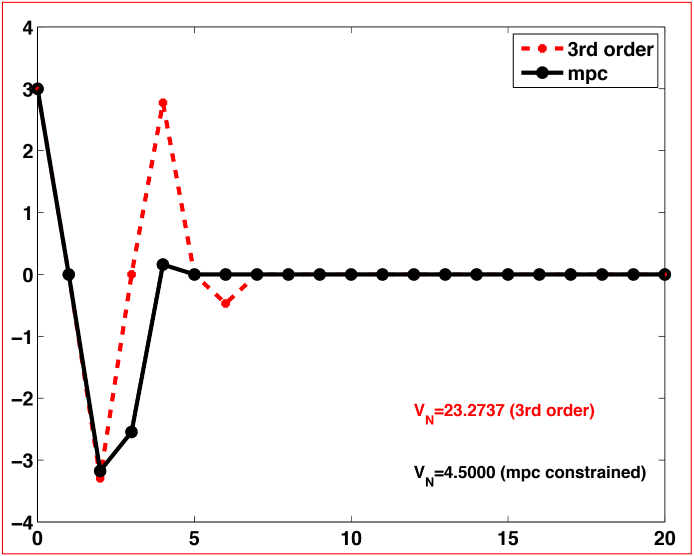
$$y(t) = x_1(t) - w_1(t),\ 0 \leq t \leq 20,\ T = 4,\ |u_{\mathsf{mpr}}(t)| \leq 90$$



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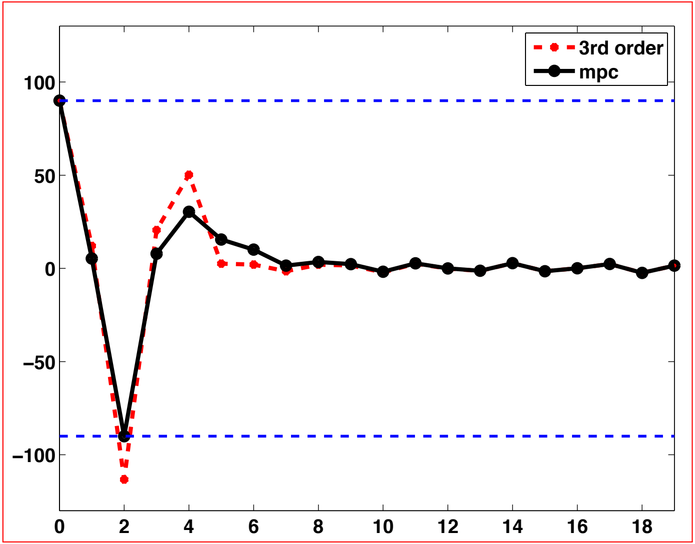

Closing Remark

We do not necessrily need $w(t)$ to come from an exosystem to do MPR. All we need is to know its values far enough in the future to be able to compute the finite horizon cost.

$$\min \sum_{s=t}^{t+T-1} l(x(t), u(t), w(t)) + \pi^T(x(t+T), w(t+T))$$

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