Energy Shaping and Quasi-linearization of Mechanical Systems

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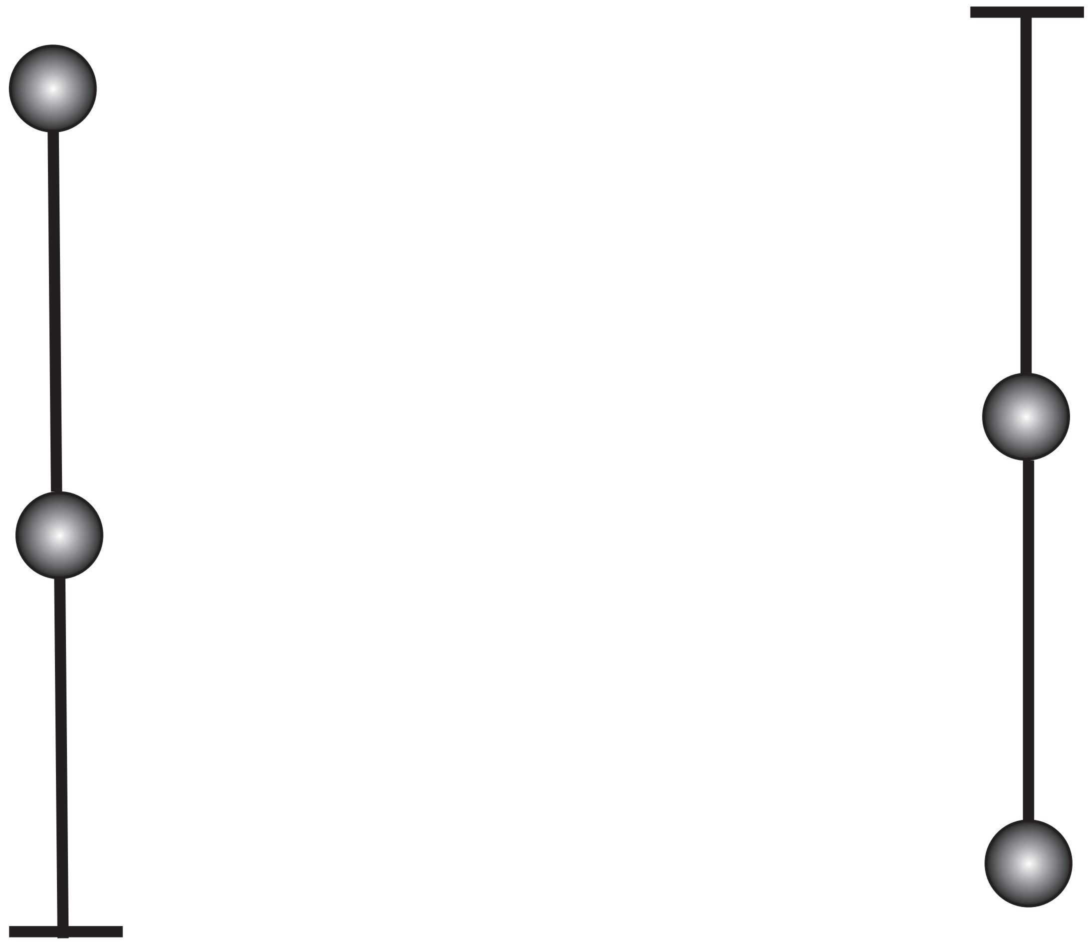
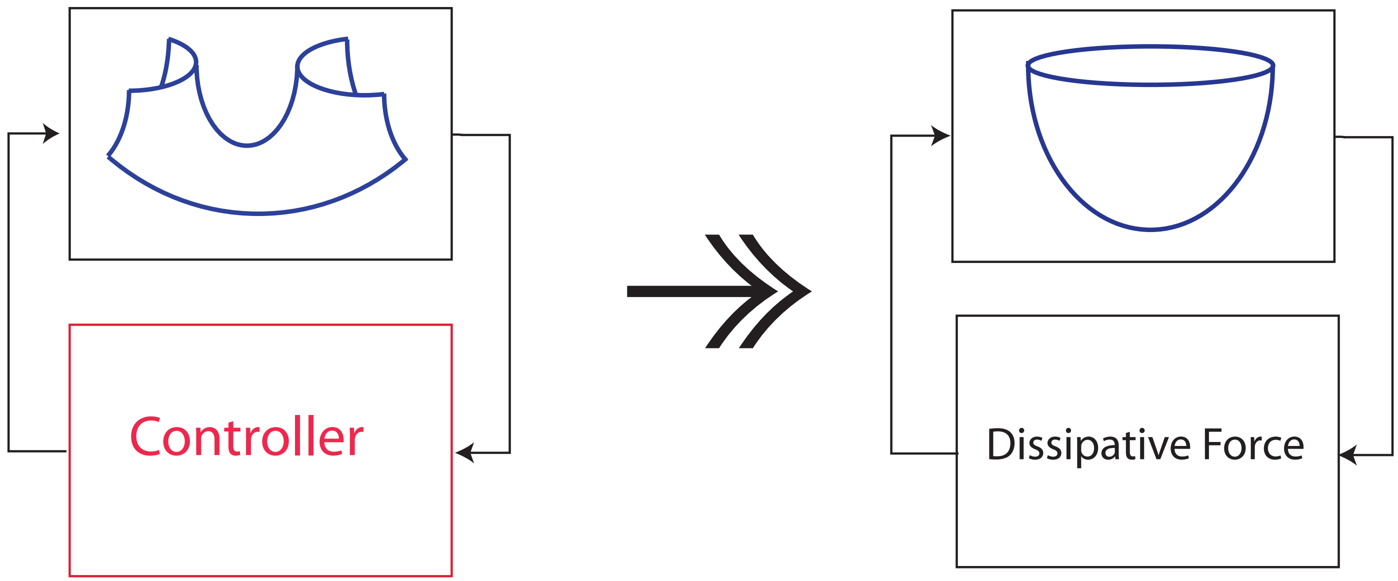
Int. Workshop on Perspectives in Dynamical Systems & Control

Motivation



Objective



Linear Energy Shaping

Example

- The unstable system

$$\ddot{x} - x = u, \quad E = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2$$

is transformed by $u = -2x - \dot{x}$ to exp. stable

$$\ddot{x} + x = -\dot{x}, \quad \widehat{E} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2.$$

- The unstable system

$$\ddot{x} - x = u, \ \ddot{y} + y = 0, \quad E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(-x^2 + y^2)$$

is transformed by $u = -2x - \dot{x}$ to Lyap. stable

$$\ddot{x} + x = -\dot{x}, \ \ddot{y} + y = 0, \quad \widehat{E} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2).$$

Example

- Impossible to shape the energy function of

$$\ddot{x} - x = u, \;\; \ddot{y} - y = 0, \quad E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(-x^2 - y^2).$$

- Any criterion for energy shapability and stabilizability by dissipation?

$$\begin{aligned}
\Sigma_1 : &\quad \ddot{x} - x = u, \\
\Sigma_2 : &\quad \ddot{x} - x = u, \quad \ddot{y} + y = 0, \\
\Sigma_3 : &\quad \ddot{x} - x = u, \quad \ddot{y} + -y = 0.
\end{aligned}$$

Controllability & Stabilizability

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m.$$

- Controllability $\Leftrightarrow \operatorname{rank}[B, AB, \cdots, A^{n-1}B] = n$
- $\dot{x} = Ax + Bu$ is controllable $\Rightarrow$ by feedback $u = -Kx$, eigenvalues of $(A - BK)$ can be arbitrarily assigned provided complex conjugates appear in pairs.
- Controllability $\Rightarrow$ stabilizability.
- Let $k = \operatorname{rank}[B, AB, \cdots, A^{n-1}B] < n$. Then, $\exists z = Px$ with $z = (z_1, z_2) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$ such that

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_{\mathrm{uc}} & 0 \\ A_{21} & A_{\mathrm{c}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_{\mathrm{c}} \end{bmatrix} u$$

where $(A_{\mathrm{c}}, B_{\mathrm{c}})$ is a controllable pair.

Controllability & Stabilizability for 2nd-Order Systems

$$\ddot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m. \tag{1}$$

- $\ddot{x} = Ax + Bu$ controllable $\Leftrightarrow$ $\dot{x} = Ax + Bu$ controllable.
- Let $k = \mathrm{rank}[B, AB, \cdots, A^{n-1}B]$. Then, $\exists z = Px$ with $z = (z_1, z_2) \in \mathbb{R}^{2(n-k)} \times \mathbb{R}^{2k}$ such that

$$\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} = \begin{bmatrix} A_{\mathrm{uc}} & 0 \\ A_{21} & A_{\mathrm{c}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_{\mathrm{c}} \end{bmatrix} u$$

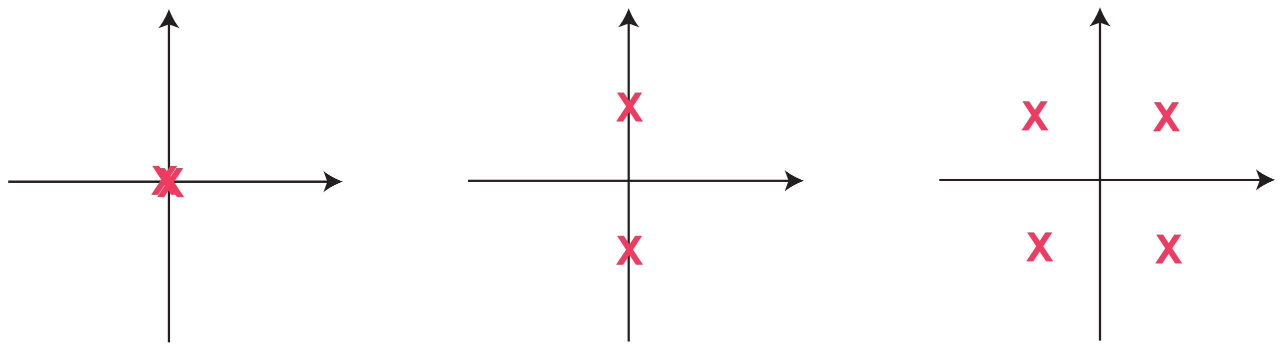
  where $(A_{\mathrm{c}}, B_{\mathrm{c}})$ is a controllable pair.
- For (1),

  controllability $\Leftrightarrow$ stabilizability.

Proof of "Controllability ⇔ Stabilizability"

$$\ddot{x} = Ax + Bu.$$

- Eigenvalues of $\ddot{x} = Ax$:



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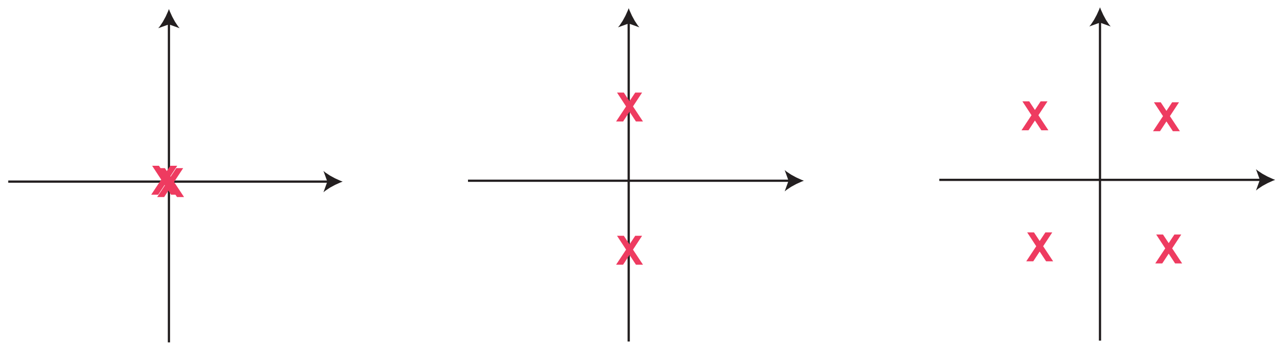

- Suppose system is uncontrollable and consider decomposition:

$$\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} = \begin{bmatrix} A_{\text{uc}} & 0 \\ A_{21} & A_{\text{c}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_{\text{c}} \end{bmatrix} u.$$

- The uncontrollable dynamics $\ddot{z}_1 = A_{\text{uc}} z_1$ is of 2nd order, so it cannot be Hurwitz. Hence, system is unstabilizable.

Stabilizability of Stable Mech. Systems by Dissipation

Consider a *stable* mechanical system with control u

$$M\ddot{x} + Sx = Bu$$

where $M = M^T \succ 0$ and $S = S^T \succ 0$.

The following are equivalent:

1. The system is controllable.
2. The system is stabilizable.
3. For any (dissipative) feedback control

   $$u = -DB^T\dot{x}, \quad D = D^T \succ 0,$$

   the closed-loop system

   $$M\ddot{x} + BDB^T\dot{x} + Sx = 0$$

   is exponentially stable.

Proof of 1 $\Rightarrow$ 3

Suppose $\lambda \in \mathbb{C}$ satisfies

$$|\lambda^2 M + \lambda BDB^T + S| = 0.$$

Then, $\exists v \neq 0 \in \mathbb{C}^n$ such that

$$v^*(\lambda^2 M + \lambda BDB^T + S) = 0.$$

Post-multiplying by v,

$$a\lambda^2 + b\lambda + c = 0$$

where $a = v^* M v > 0$, $b = v^* BDB^T v \geq 0$, $c = v^* S v > 0$.

$$\begin{aligned} b = 0 \Rightarrow v^* B = 0, &\quad v^*(\lambda^2 M + S) = 0 \\ &\Rightarrow v^*[\lambda^2 M + S, B] = 0 \\ &\Rightarrow \text{rank}[\lambda^2 M + S, B] < n \\ &\Rightarrow \text{uncontrollable (why? use PBC test).} \end{aligned}$$

Hence, $b > 0$, and thus $\text{Re}[\lambda] < 0$, implying exponential stability.

Oscillatory Dynamics

Definition

1st-order system $\dot{x} = Ax$ is called **oscillatory** if A is diagonalizable and each $\lambda(A)$ is a non-zero purely imaginary number.

Example

Two (1st and 4th) oscillatory and three non-oscillatory systems:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}$$

Theorem

$\ddot{x} = Ax$ is oscillatory $\Leftrightarrow \exists\ M = M^T \succ 0$ and $S = S^T \succ 0$ such that $A = -M^{-1}S$. In other words,

$$\ddot{x} = Ax \quad \Leftrightarrow \quad \ddot{x} = -M^{-1}Sx \quad \Leftrightarrow \quad M\ddot{x} + Sx = 0.$$

Energy Shaping of Linear Mechanical Systems

Linear mechanical system:

$$\Sigma : \qquad M\ddot{q} + Sq = Bu,$$

where $M = M^T \succ 0$, $S = S^T$, $q \in \mathbb{R}^n$.
Objective 1 (Energy Shaping): Find position feedback $u = -Kq + v$ to transform Σ to a stable mechanical system

$$\widehat{\Sigma} : \qquad \widehat{M}\ddot{q} + \widehat{S}q = \widehat{B}v$$

where

$$\widehat{M} = \widehat{M}^T \succ 0, \quad \widehat{S} = \widehat{S}^T \succ 0.$$

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Answer: It is possible $\Leftrightarrow$ Σ is controllable or its uncontrollable dynamics is oscillatory.

$$\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} = \begin{bmatrix} A_{\mathrm{uc}} & 0 \\ A_{21} & A_{\mathrm{c}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_{\mathrm{c}} \end{bmatrix} u.$$

Energy Shaping of Linear Mechanical Systems

Linear mechanical system:

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where $M = M^T \succ 0$, $S = S^T$, $q \in \mathbb{R}^n$.

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$$\widehat{\Sigma} : \qquad \widehat{M}\ddot{q} + \widehat{S}q = \widehat{B}v$$

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Objective 2 (Stabilization by Dissipation): Can dissipative feedback $v = -D\widehat{B}^T\dot{q}$ exponentially stabilize the energy-shaped system $\widehat{\Sigma}$?

Energy Shaping of Linear Mechanical Systems

Linear mechanical system:

$$\Sigma : \qquad M\ddot{q} + Sq = Bu,$$

where $M = M^T \succ 0$, $S = S^T$, $q \in \mathbb{R}^n$.

Objective 1 (Energy Shaping): Find position feedback $u = -Kq + v$ to transform Σ to a stable mechanical system

$$\widehat{\Sigma} : \qquad \widehat{M}\ddot{q} + \widehat{S}q = \widehat{B}v$$

where

$$\widehat{M} = \widehat{M}^T \succ 0, \quad \widehat{S} = \widehat{S}^T \succ 0.$$

Answer: It is possible $\Leftrightarrow$ Σ is controllable or its uncontrollable dynamics is oscillatory.

$$\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} = \begin{bmatrix} A_{\mathrm{uc}} & 0 \\ A_{21} & A_{\mathrm{c}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_{\mathrm{c}} \end{bmatrix} u.$$

Objective 2 (Stabilization by Dissipation): Can dissipative feedback $v = -D\widehat{B}^T\dot{q}$ exponentially stabilize the energy-shaped system $\widehat{\Sigma}$?

Answer: Yes $\Leftrightarrow$ Σ is controllable.

Summary

	controllable	oscillatory uncontrollable dynamics	non-oscillatory uncontrollable dynamics
energy shapability	Yes	Yes	No
stabilizability by dissipation after energy shaping	Yes	No	N/A

- not energy-shapable

$$\ddot{x} - x = 0, \quad \ddot{y} - y = u$$

- energy-shapable, but not stabilizable by dissipation after shaping

$$\ddot{x} + x = 0, \quad \ddot{y} - y = u$$

- energy shapable, and stabilizable by dissipation after shaping

$$\ddot{x} + x + y = 0, \quad \ddot{y} + x - y = u$$

Non-Linear Energy Shaping

"Simple" Mechanical Systems

- Lagrangian $L(q,\dot{q}) = K(q,\dot{q}) - V(q) = \frac{1}{2} m(q)_{ij}\dot{q}^i\dot{q}^j - V(q)$.
- Total Energy: $E(q,\dot{q}) = K(q,\dot{q}) + V(q)$
- Force: $F = (F_1, \cdots, F_n) \in \mathbb{R}^n$ (actually, T^*Q-valued).
- Equations of motion:

$$\begin{aligned} & \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = F_i, \quad i = 1, \ldots, n \\ \Leftrightarrow \quad & m_{ij}\ddot{q}^j + [jk,i]\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^i} = F_i, \quad i = 1, \ldots, n \\ \Leftrightarrow \quad & \ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k + m^{ij}\frac{\partial V}{\partial q^j} = m^{ij}F_j, \quad i = 1, \ldots, n \end{aligned}$$

where

$$\begin{aligned} [jk,i] &= \frac{1}{2}\left(\frac{\partial m_{ik}}{\partial q^j} + \frac{\partial m_{ji}}{\partial q^k} - \frac{\partial m_{jk}}{\partial q^i}\right), \\ \Gamma^i_{jk} &= m^{i\ell}[jk,\ell]. \end{aligned}$$

Force Types

Total energy is $E(q,\dot q) = K(q,\dot q) + V(q)$. Along the trajectory of the system we have

$$\frac{dE}{dt} = \langle F, \dot q\rangle$$

rate of change in energy = power.

Normally, $F : TQ \to T^*Q$, i.e., $F = F(q,\dot q)$.

- F is called **dissipative** if

$$\langle F(q,\dot q), \dot q\rangle \le 0 \quad \forall (q,\dot q) \in TQ.$$

- F is called **gyroscopic** if

$$\langle F(q,\dot q), \dot q\rangle = 0 \quad \forall (q,\dot q) \in TQ.$$

- F is called **locally dissipative** if $\langle F(q,\dot q), \dot q\rangle \le 0$ for all $(q,\dot q)$ in neighborhood of $(0,0)$.

Gyroscopic Force Quadratic in Velocity

$$F(q,\dot{q}) = \begin{bmatrix} C_{ij1}(q)\dot{q}^i\dot{q}^j \\ \vdots \\ C_{ijn}(q)\dot{q}^i\dot{q}^j \end{bmatrix}, \qquad C_{ijk} = C_{jik}.$$

Theorem

1. Quadratic dissipative force = quadratic gyroscopic force.

2. F is gyroscopic force iff

$$C_{ijk} = C_{jik}, \quad C_{ijk} + C_{jki} + C_{kij} = 0.$$

Proof.

$\langle F(q,\dot{q}), \dot{q}\rangle = C_{ijk}\dot{q}^i\dot{q}^j\dot{q}^k \leq 0$. Being cubic in $\dot{q}$,

$$C_{ijk}\dot{q}^i\dot{q}^j\dot{q}^k = 0.$$

Hence, $\mathrm{Sym}(C_{ijk}) = 0$, or

$$C_{ijk} + C_{jki} + C_{kij} + C_{ikj} + C_{kji} + C_{jik} = 0.$$

Not-So-Good Tradition in Robotics

- Many robotics books write equations of motion in the following form

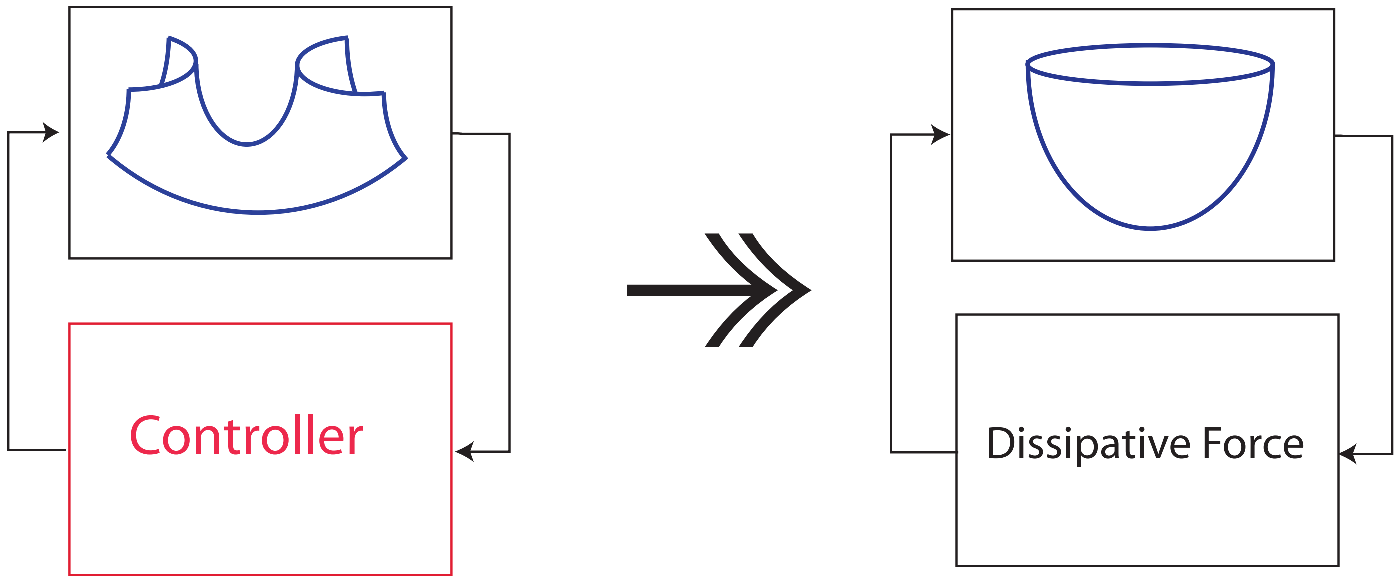
$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + dV = F$$

  and then claim that

$$\dot{M} - 2C$$

  is a skew-symmetric matrix. They then state that this property implies energy conservation when $F = 0$.
- However, this skew-symmetric property is of little use. One can show energy conservation very simply without this observation of skew-symmetric property of the strange quantity "$\dot{M} - 2C$."
- This unfortunate tradition comes from the unnecessary effort to put a (0,3) tensor into "matrix form".

Objective & Strategy



$$\Sigma \xrightarrow{\text{Energy-shape by solving PDEs}} \widehat{\Sigma}$$

Linearize: $\Sigma \downarrow \Sigma^{\ell}$

Initial data: $\widehat{\Sigma}^{\ell} \uparrow \widehat{\Sigma}$

$$\Sigma^{\ell} \xrightarrow{\text{Energy-shape with linear algebra}} \widehat{\Sigma}^{\ell}$$

Matching

$$\underbrace{\underbrace{1,\ldots,n_1}_{\alpha,\beta,\gamma,\ldots};\underbrace{n_1+1,\ldots,n}_{a,b,c,\ldots}}_{i,j,k\ldots}$$

Given cL system with $L = \frac{1}{2}m_{ij}\dot{q}^i\dot{q}^j - V(q)$ with $(n-n_1)$ control u_a:

$$\begin{cases} m_{\alpha j}\ddot{q}^j + [jk,\alpha]\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^\alpha} = 0 \\ m_{aj}\ddot{q}^j + [jk,a]\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^a} = u_a \end{cases} \tag{2}$$

find a feedback equivalent cL system with $\widehat{L} = \frac{1}{2}\widehat{m}_{ij}\dot{q}^i\dot{q}^j - \widehat{V}$:

$$\widehat{m}_{ij}\ddot{q}^j + \widehat{[jk,i]}\dot{q}^j\dot{q}^k + \frac{\partial\widehat{V}}{\partial q^i} = \widehat{C}_{jki}\dot{q}^j\dot{q}^k + \widehat{u}_i \tag{3}$$

where $\widehat{u}\in\widehat{W}$ of $\dim\widehat{W} = n-n_1$, $\widehat{[jk,i]}$ are Christoffel symbols of $\widehat{m}$, and

$$\widehat{C}_{ijk} = \widehat{C}_{jik}, \quad \widehat{C}_{ijk} + \widehat{C}_{jki} + \widehat{C}_{kij} = 0.$$

Matching Conditions

Solving (3) for $\ddot{q}^j$, substituting them into (2), and collecting terms of equal degrees in $\dot{q}$

$$\text{kinetic matching}: \quad m_{\alpha k}\widehat{m}^{kl}\left(\widehat{[ij,l]} - \widehat{C}_{ijl}\right) - [ij,\alpha] = 0, \tag{4}$$

$$\text{potential matching}: \quad m_{\alpha k}\widehat{m}^{kl}\frac{\partial \widehat{V}}{\partial q^l} - \frac{\partial V}{\partial q^\alpha} = 0, \tag{5}$$

$$\text{control bundle matching}: \quad \left\langle \widehat{W}, m_{\alpha k}\widehat{m}^{kl}\frac{\partial}{\partial q^l}\right\rangle = 0 \tag{6}$$

and

$$u_a = [jk,a]\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^a} - m_{ar}\widehat{m}^{rl}\left(\widehat{[jk,l]}\dot{q}^j\dot{q}^k + \frac{\partial \widehat{V}}{\partial q^l} - \widehat{C}_{jkl}\dot{q}^j\dot{q}^k - \widehat{u}_l\right).$$

where # of PDE's in kinetic matching = $\frac{n_1 n(n+1)}{2}$.
Hence, need to solve PDEs for $(\widehat{m}_{ij}) = (\widehat{m}_{ji}) \succ 0$, $\widehat{V}$ with $D^2\widehat{V}(q_e) \succ 0$, and gyroscopic $\widehat{C}_{ijk}$. Then $\widehat{W}$ is uniquely determined as
$\widehat{W} = \widehat{m}m^{-1}\,\mathrm{span}\{dq^a\} = \mathrm{span}\{m^{ai}\widehat{m}_{ij}dq^j\}$.

Decomposition and Reduction of Kinetic Matching PDEs

Decompose the $\frac{n_1 n(n+1)}{2}$ kinetic matching PDEs

$$m_{\alpha k}\widehat{m}^{kl}\left(\widehat{[ij,l]} - \widehat{C}_{ijl}\right) - [ij,\alpha] = 0$$

into two sets: one without $\widehat{C}_{ijk}$ and the other with $\widehat{C}_{ijk}$. Let

$$\widehat{A}_{ijk} = m_{ip}m_{jq}m_{kr}\widehat{m}^{pl}\widehat{m}^{qs}\widehat{m}^{rt}\widehat{C}_{lst}, \tag{7}$$

$$\widehat{S}_{ijk} = m_{ip}m_{jq}\widehat{m}^{pl}\widehat{m}^{qs}\left(m_{kr}\widehat{m}^{rt}\widehat{[ls,t]} - [ls,k]\right). \tag{8}$$

The kinetic matching (4) is equivalent to

$$\widehat{A}_{ij\alpha} = \widehat{S}_{ij\alpha}. \tag{9}$$

Write the Jacobi identities for $\widehat{A}_{ijk}$ in the following four sets of equations:

$$\widehat{A}_{\alpha\beta\gamma} + \widehat{A}_{\beta\gamma\alpha} + \widehat{A}_{\gamma\alpha\beta} = 0, \tag{10}$$

$$\widehat{A}_{a\beta\gamma} + \widehat{A}_{\beta\gamma a} + \widehat{A}_{\gamma a\beta} = 0, \tag{11}$$

$$\widehat{A}_{ab\gamma} + \widehat{A}_{b\gamma a} + \widehat{A}_{\gamma ab} = 0, \tag{12}$$

$$\widehat{A}_{abc} + \widehat{A}_{bca} + \widehat{A}_{cab} = 0. \tag{13}$$

Decomposition and Reduction of Kinetic Matching PDE's

By (9), eqns (10) – (13) are equivalent to

$$\widehat{S}_{\alpha\beta\gamma} + \widehat{S}_{\beta\gamma\alpha} + \widehat{S}_{\gamma\alpha\beta} = 0, \tag{14}$$

$$\widehat{S}_{a\beta\gamma} + \widehat{A}_{\beta\gamma a} + \widehat{S}_{\gamma a\beta} = 0, \tag{15}$$

$$\widehat{S}_{ab\gamma} + \widehat{A}_{b\gamma a} + \widehat{A}_{\gamma ab} = 0, \tag{16}$$

$$\widehat{A}_{abc} + \widehat{A}_{bca} + \widehat{A}_{cab} = 0. \tag{17}$$

where

$$\widehat{S}_{ijk} = m_{ip}m_{jq}\widehat{m}^{pl}\widehat{m}^{qs}\left(m_{kr}\widehat{m}^{rt}\frac{1}{2}\left(\frac{\partial \widehat{m}_{ts}}{\partial q^l} + \frac{\partial \widehat{m}_{tl}}{\partial q^s} - \frac{\partial \widehat{m}_{ls}}{\partial q^t}\right) - [ls,k]\right).$$

1. Solve PDEs (14) for $\widehat{m}_{ij}$ where # of PDEs in (14) $= \frac{n_1(n_1+1)(n_1+2)}{6} \leq \frac{n_1 n(n+1)}{2}$ = # of orignal kinetic PDEs in (4). ('=' holds iff $n_1 = n - 1$, i.e., underactuation degree 1)
2. $\widehat{A}_{ij\alpha} = \widehat{S}_{ij\alpha}$ from (9).
3. $\widehat{A}_{\beta\gamma a} = -\widehat{S}_{a\beta\gamma} - \widehat{S}_{\gamma a\beta}$ and $\widehat{A}_{\gamma ab} = \widehat{A}_{b\gamma a} = -\frac{1}{2}\widehat{S}_{ab\gamma}$ from (15) and (16).
4. Choose any $\widehat{A}_{abc}$'s such that (17) holds. E.g. $\widehat{A}_{abc} = 0$.
5. Compute $\widehat{C}_{ijk}$ from (7).

Further "Reduction" of Total Matching PDEs

- Total $(\frac{n_1(n_1+1)(n_1+2)}{6}+n_1)$ PDE's for $(\frac{n(n+1)}{2}+1)$ unknowns, $\widehat{m}_{ij}$ and $\widehat{V}$:

$$\widehat{S}_{\alpha\beta\gamma}+\widehat{S}_{\beta\gamma\alpha}+\widehat{S}_{\gamma\alpha\beta}=0; \qquad m_{\alpha k}\widehat{m}^{kl}\frac{\partial\widehat{V}}{\partial q^l}-\frac{\partial V}{\partial q^\alpha}=0,$$

where

$$\widehat{S}_{\alpha\beta\gamma}=m_{\alpha p}m_{\beta q}\widehat{m}^{pl}\widehat{m}^{qs}\left(m_{\gamma r}\widehat{m}^{rt}\frac{1}{2}\left(\frac{\partial\widehat{m}_{ts}}{\partial q^l}+\frac{\partial\widehat{m}_{tl}}{\partial q^s}-\frac{\partial\widehat{m}_{ls}}{\partial q^t}\right)-[ls,\gamma]\right).$$

- Let $\widehat{T}=m\widehat{m}^{-1}m$, so that finding $\widehat{T}\Leftrightarrow$ finding $\widehat{m}$. The, matching PDEs become
- Total $(\frac{n_1(n_1+1)(n_1+2)}{6}+n_1)$ PDEs for $(\frac{n_1(2n-n_1+1)}{2}+1)$ unknowns, $\widehat{T}_{\alpha i}$ and $\widehat{V}$:

$$\widehat{J}_{\alpha\beta\gamma}+\widehat{J}_{\beta\gamma\alpha}+\widehat{J}_{\gamma\alpha\beta}=0; \qquad \widehat{T}_{\alpha i}m^{il}\frac{\partial\widehat{V}}{\partial q^l}-\frac{\partial V}{\partial q^\alpha}=0,$$

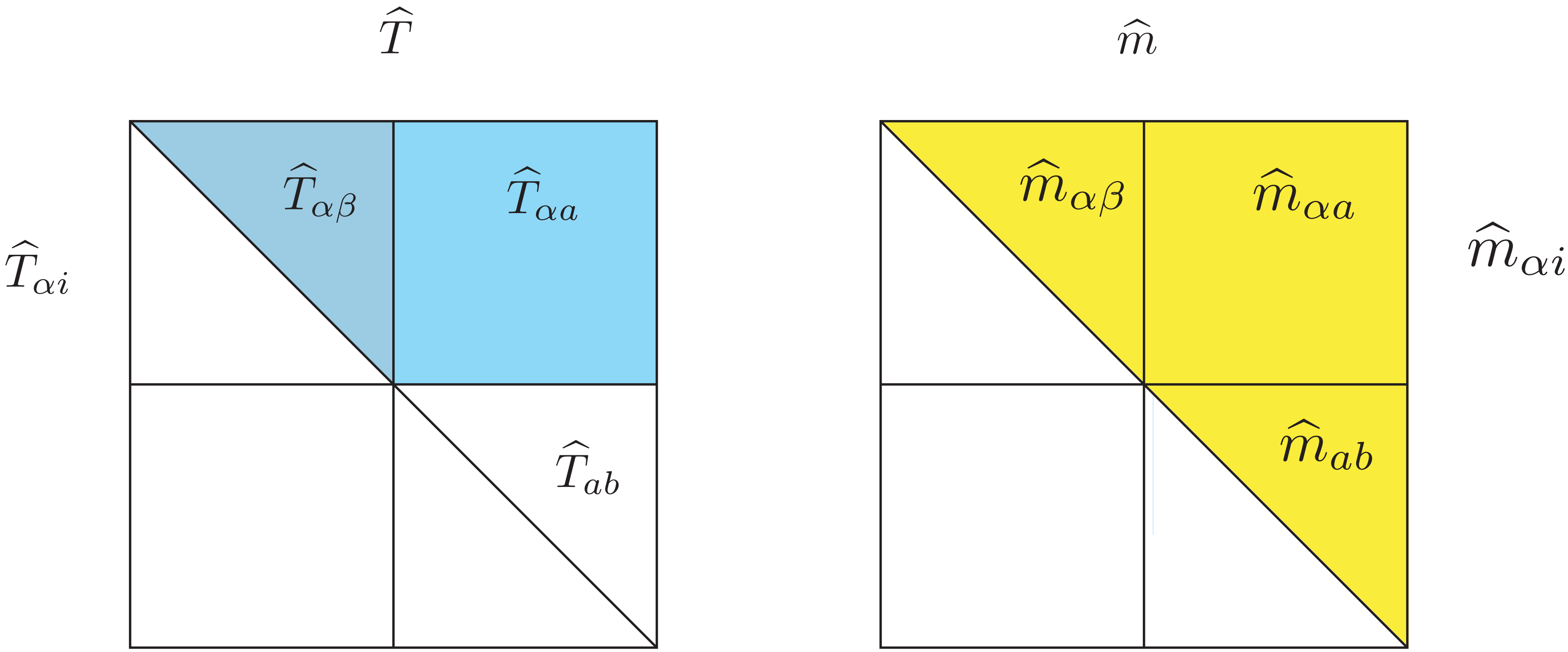
where

$$\widehat{J}_{\alpha\beta\gamma}=\frac{1}{2}\widehat{T}_{\gamma s}m^{sk}\left(\frac{\partial\widehat{T}_{\alpha\beta}}{\partial q^k}-\widehat{T}_{\alpha i}\Gamma^i_{\beta k}-\widehat{T}_{\beta i}\Gamma^i_{\alpha k}\right).$$

Superiority of use of $\widehat{T}$ to that of $\widehat{m}$

$\widehat{T}$	$\widehat{m}$
$\widehat{J}_{\alpha\beta\gamma} + \widehat{J}_{\beta\gamma\alpha} + \widehat{J}_{\gamma\alpha\beta} = 0,$ $\widehat{J}_{\alpha\beta\gamma} = \frac{1}{2}\widehat{T}_{\gamma s} m^{sk}\left(\frac{\partial \widehat{T}_{\alpha\beta}}{\partial q^k} - \widehat{T}_{\alpha i}\Gamma^i_{\beta k} - \widehat{T}_{\beta i}\Gamma^i_{\alpha k}\right)$	$\widehat{S}_{\alpha\beta\gamma} + \widehat{S}_{\beta\gamma\alpha} + \widehat{S}_{\gamma\alpha\beta} = 0$ $\widehat{S}_{\alpha\beta\gamma} = m_{\alpha p} m_{\beta q} \widehat{m}^{pl} \widehat{m}^{qs}\left(m_{\gamma r}\widehat{m}^{rt}\widehat{[ls,t]} - [ls,\gamma]\right)$
$\frac{n_1(2n-n_1+1)}{2}$ unknowns $\widehat{T}_{\alpha i}$	$\frac{n(n+1)}{2}$ unknowns $\widehat{m}_{ij}$
$\frac{n_1(n_1+1)}{2}n$ first-order partials $\frac{\partial \widehat{T}_{\alpha\beta}}{\partial q^k}$	$\frac{n(n+1)}{2}n$ first-order partials $\frac{\partial \widehat{m}_{ij}}{\partial q^k}$

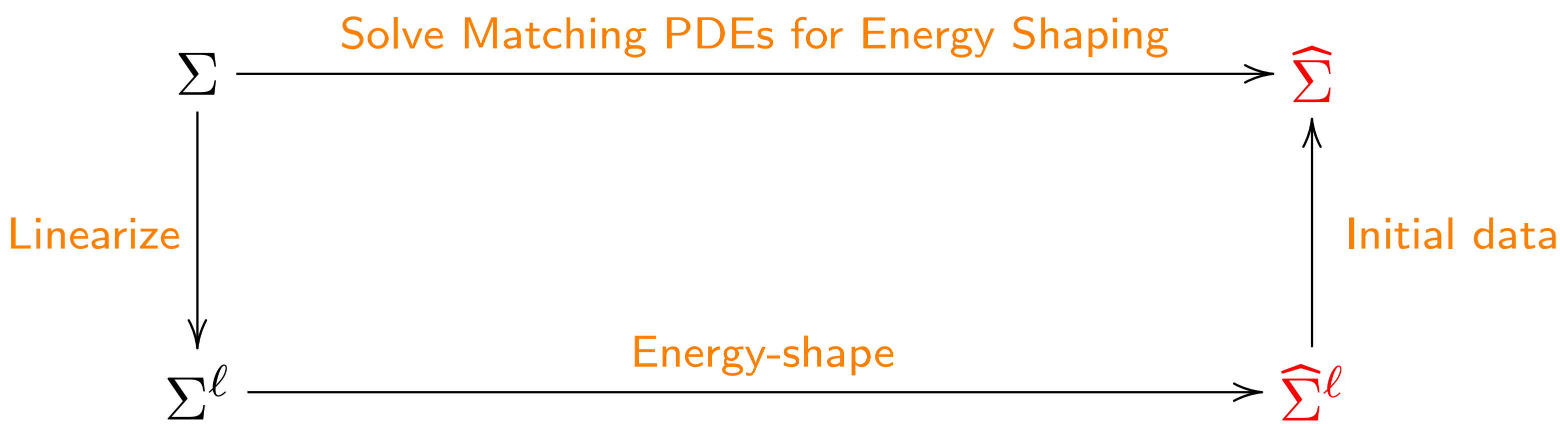
Illustration: Superiority of $\widehat{T}$ to $\widehat{m}$



Energy Shaping for Systems with Underactuation Degree 1

2 Matching PDEs for $\widehat{V}$ and $\widehat{T}_{11}, \ldots, \widehat{T}_{1n}$ (and $\widehat{T}_{ab}$, $2 \le a, b \le n$).

$$\widehat{T}_{1j} m^{jk}\left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\widehat{T}_{1i}\Gamma^i_{1k}\right) = 0; \qquad \widehat{T}_{1i} m^{il}\frac{\partial \widehat{V}}{\partial q^l} - \frac{\partial V}{\partial q^1} = 0.$$



Theorem (Energy Shaping)

Σ is energy-shapable

$\Leftrightarrow$ its linearization Σ^ℓ is energy shapable

$\Leftrightarrow$ Σ^ℓ is controllable or its uncontrollable dynamics is oscillatory.

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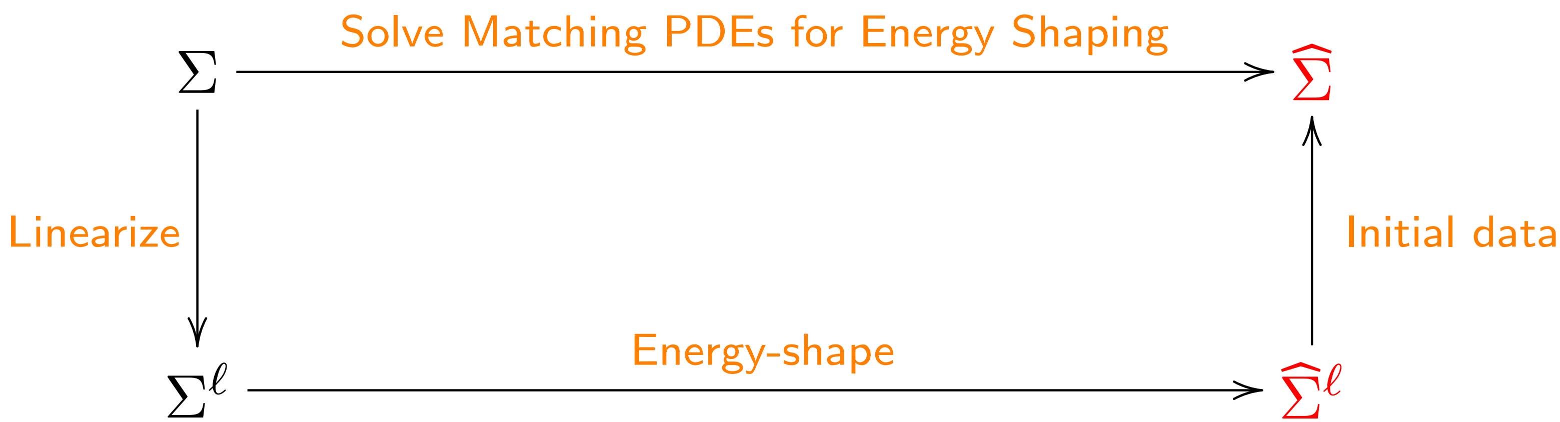


Theorem (Energy Shaping)

Σ is energy-shapable

$\Leftrightarrow$ its linearization Σ^ℓ is energy shapable. (no hat here!)

$\Leftrightarrow$ Σ^ℓ is controllable or its uncontrollable dynamics is oscillatory.(no hat here!)

Theorem (Stabilization by Dissipation after Energy Shaping)

Energy-shaped $\widehat{\Sigma}$ is exp. stabilized. by any linear dissipative feedback of full rank

$\Leftrightarrow$ the linearization Σ^ℓ of the original system Σ is controllable. (no hat here!)

Example: PVTOL

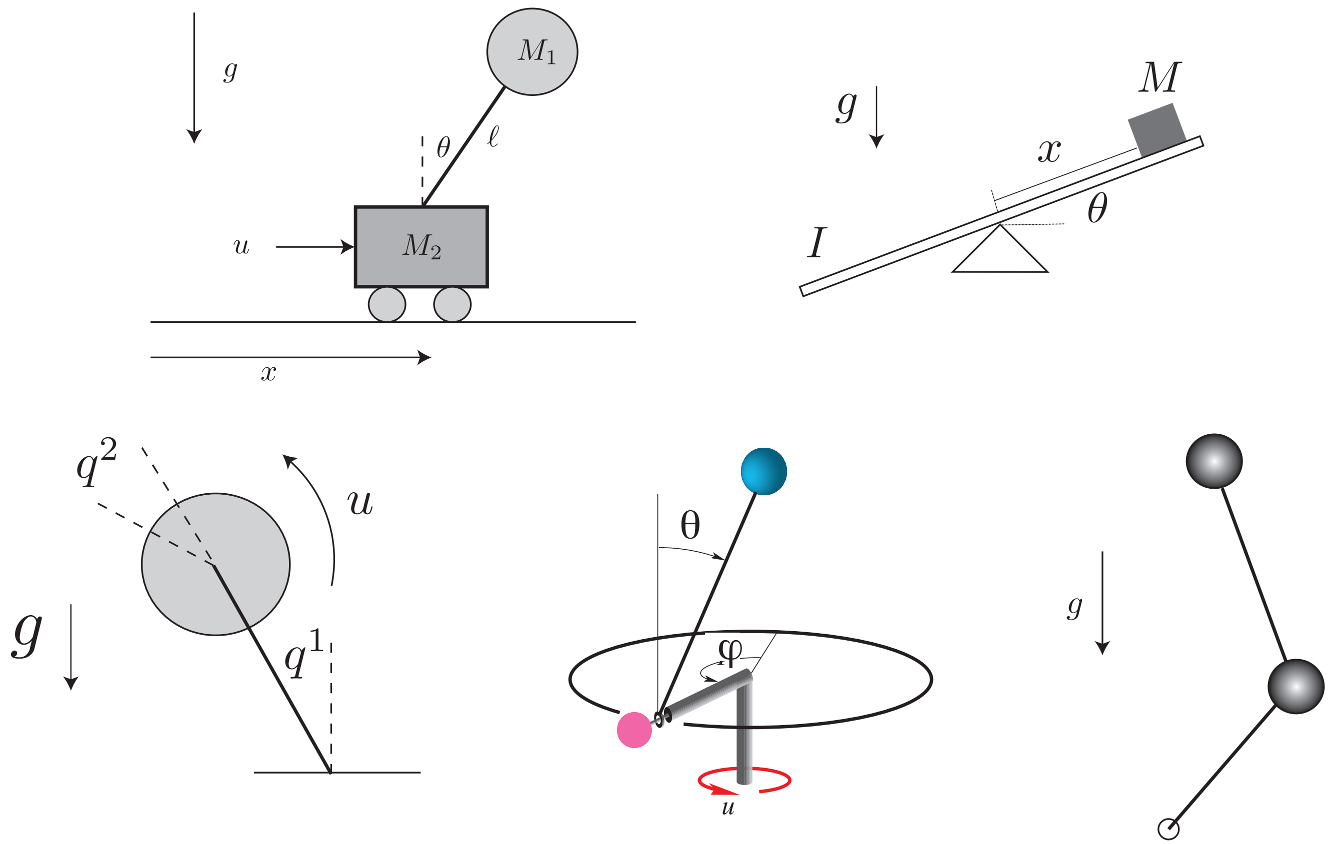
Controlled Lagrangian dynamics of a planar vertical takeoff and landing (PVTOL) aircraft

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{g}{c}\sin\theta \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{\epsilon}\cos\theta & \frac{1}{\epsilon}\sin\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $g, c > 0$, $\epsilon \neq 0$, and $q = (x, y, \theta) \in \mathbb{R}^3$.

- Equilibria $(x_e, y_e, 0)$.
- Degree of under-actuation $= 1$.
- Linearization at $(x_e, y_e, 0)$ is controllable.
- Therefore, it can be energy-shaped and then be exponentially stabilized by any linear symmetric dissipative feedback force of full rank.

More Examples



Linearization of each is controllable, so exponential stabilization by energy shaping + dissipation is possible. See Ng, Chang and Song[2013] for detailed computation.

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Quasi-Linearization of Mechanical Systems

Equations of Motion of Mechanical System

Lagrangian

$$L(x, \dot{x}) = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V(x).$$

Equations of Motion:

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k + g^{ij} \partial_j V = 0$$

or

$$\frac{d}{dt} x^i = \dot{x}^i$$
$$\frac{d}{dt} \dot{x}^i = -\Gamma^i_{jk} \dot{x}^j \dot{x}^k - g^{ij} \partial_j V.$$

$\exists$ coordinate system in which $\Gamma^i_{jk} = 0 \Leftrightarrow R = 0$.

Quasilinearization

Linear transformation of velocity $\dot{x}$:

$$(x^i, \dot{x}^i) \mapsto (x^i, v^i = A^i_j(x)\dot{x}^j) \tag{18}$$

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Equations of motion in (x, v) coordinates:

$$\dot{x}^i = B^i_j v^j,$$
$$\dot{v}^i = \frac{1}{2}\left(\partial_k A^i_j + \partial_j A^i_k - 2A^i_l \Gamma^l_{\ jk}\right)\dot{x}^j \dot{x}^k - A^i_j g^{jk}\partial_k V,$$

where B^i_j be the inverse of A^i_j, i.e., $B^i_j A^j_k = \delta^i_k$.

Quasilinearization

Linear transformation of velocity $\dot{x}$:

$$(x^i, \dot{x}^i) \mapsto (x^i, v^i = A^i_j(x)\dot{x}^j) \tag{18}$$

Equations of motion in (x, v) coordinates:

$$\begin{aligned} \dot{x}^i &= B^i_j v^j, \\ \dot{v}^i &= \frac{1}{2}\left(\partial_k A^i_j + \partial_j A^i_k - 2A^i_l \Gamma^l_{jk}\right)\dot{x}^j \dot{x}^k - A^i_j g^{jk} \partial_k V, \end{aligned}$$

where B^i_j be the inverse of A^i_j, i.e., $B^i_j A^j_k = \delta^i_k$.
Equations of motion become

$$\begin{aligned} \dot{x}^i &= B^i_j v^j, \\ \dot{v}^i &= \qquad - A^i_j g^{jk} \partial_k V, \end{aligned}$$

if and only if

$$\partial_k A^i_j + \partial_j A^i_k - 2A^i_\ell \Gamma^\ell_{jk} = 0. \tag{19}$$

Quasilinearizability in terms of Killing Vector Fields

A vector field $X = X^i \partial_i$ on a Riemannian manifold (M, g) is called a **Killing (vector) field** if it satisfies the **Killing equation**

$$L_X g = 0$$

or in coordinates

$$\partial_k \alpha_j + \partial_j \alpha_k - 2\alpha_\ell \Gamma^\ell_{jk} = 0,$$

where $\alpha = g^\flat X = g_{jk} X^k dx^j$.

Theorem

Quasilinearizability:

$$\partial_k A^i_j + \partial_j A^i_k - 2A^i_\ell \Gamma^\ell_{jk} = 0$$

$\Leftrightarrow$ *existence of n linearly independent Killing fields ($\mathfrak{iso}(M,g)_p = T_p M$).*

Sufficient Conditions for Quasilinearizability

Theorem

Let p be a point in (M, g).

1. Quasilinearization is possible around $p \in M$ if $\nabla R = 0$ in a neighborhood of p (i.e., local symmetricity).

2. Suppose $\dim M = 2$. Then, quasilinearization is possible around $p \in M$ if and only if the scalar curvature R_S of g is constant in a neighborhood of p.

Remark:

- Easy to verify by differentiation only (c.f. Venkatraman, Ortega, Sarras, and van der Schaft [2010]).
- More general than the condition $R = 0$ that was independently made use of by Bedrossian [1992] and Spring [1992].

Integrability Conditions of Killing Equation [Yano]

The Killing equation and all of its integrability conditions constitute the following involutive system of PDEs:

$$\begin{cases} L_X g = 0 \\ L_X \nabla = 0 \\ L_X R = 0, \\ L_X(\nabla^k R) = 0, \quad k = 1, 2, 3, \ldots. \end{cases}$$

or

$$\begin{cases} g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0 \\ (\nabla^2 X)(Y, Z) + R(X, Y)Z = 0 \\ (\nabla_X R)(Y, Z)U - \nabla_{R(Y,Z)U} X + R(\nabla_Y X, Z)U + R(Y, \nabla_Z X)U + R(Y, Z)\nabla_U X = 0 \\ L_X(\nabla^k R) = 0, \quad k = 1, 2, 3, \ldots. \end{cases}$$

for all $Y, Z, U \in \mathfrak{X}(M)$.

The map $X \in \mathfrak{iso}(M, g) \mapsto (X|_p, (\nabla X)|_p)$ is 1-1 and linear.

- $\nabla R = 0 \Rightarrow \mathfrak{iso}(M, g)(p) = T_p M$.
- $R_S = const. \Leftrightarrow \mathfrak{iso}(M, g)(p) = T_p M$ for $\dim M = 2$.

Mechanical Meaning of Quasilinearizability

For a Lagrangian $L = \frac{1}{2} g(\dot{x}, \dot{x})$,

$$\partial_k \alpha_j + \partial_j \alpha_k - 2\alpha_\ell \Gamma^\ell_{jk} = 0$$

$\Leftrightarrow$

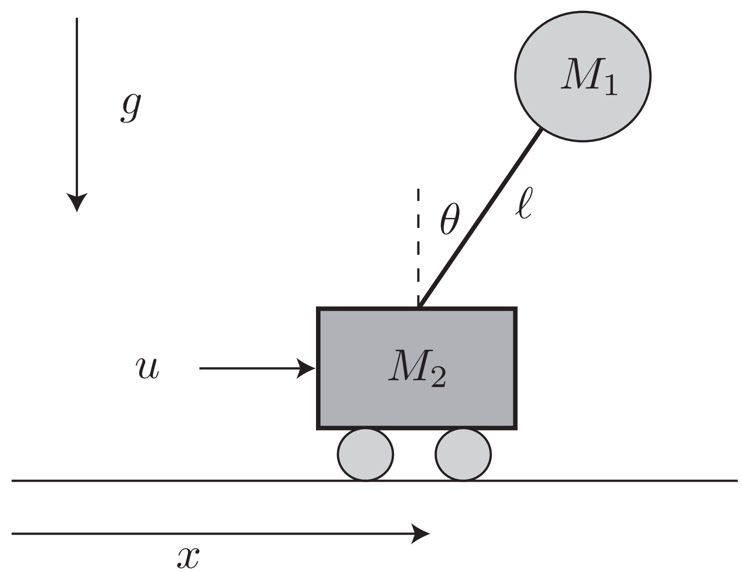
$$\alpha_i \dot{x}^i(t) = \text{constant in } t.$$

Namely, quasilinearizability is equivalent to the existence of n independent first integrals that are linear in the velocity.

For example, angular momentum conservation in the free rigid body dynamics implies quasilinearizability. Indeed,

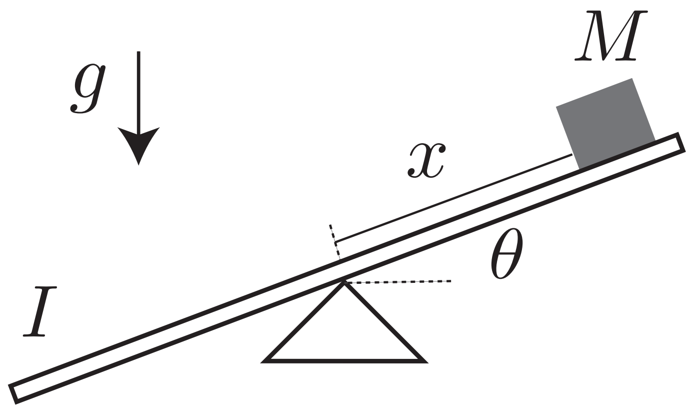
$$\dot{\mathbf{R}} = \mathbf{R}(\mathbb{I}^{-1}\mathbf{R}^{-1}\pi)^\wedge$$
$$\dot{\pi} = 0_3.$$

Inverted Pendulum on a Cart



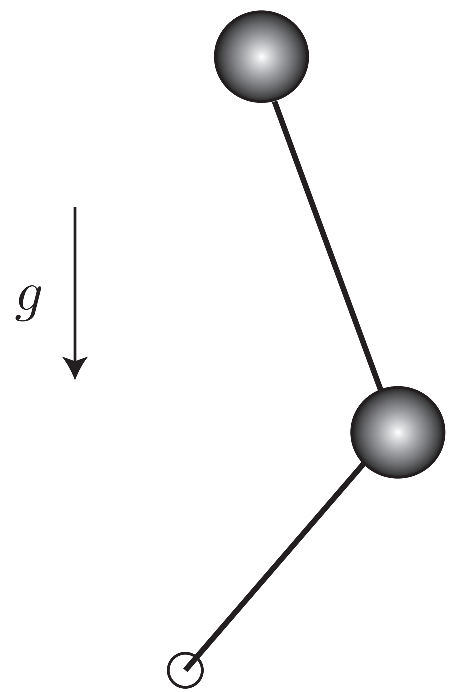
Scalar curvature $R_S = 0 \Rightarrow$ quasilinearizable.

Mass and Beam



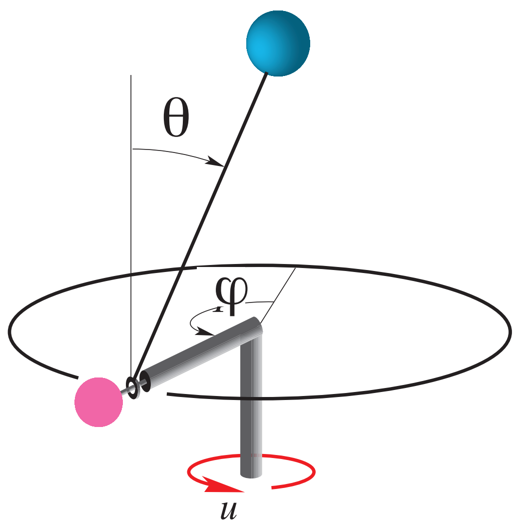
Scalar curvature $R_S = \frac{2I}{(Mx^2+1)^2}$ is not constant $\Rightarrow$ NOT quasilinearizable.

Pendubot



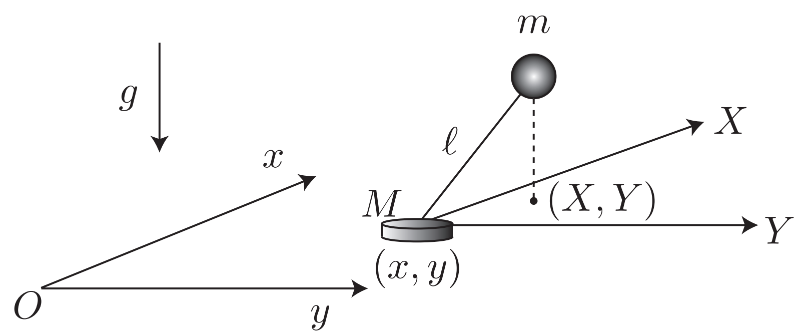
non-constant scalar curvature $\Rightarrow$ NOT quasilinearizable.

Furuta Pendululm



non-constant scalar curvature ⇒ NOT quasilinearizable.

Spherical Pendulum on a puck



$\mathfrak{iso}(M, g)$ is generated by

$$X_1 = \frac{\partial}{\partial x}$$

$$X_2 = \frac{\partial}{\partial y}$$

$$X_3 = Y\frac{\partial}{\partial X} - X\frac{\partial}{\partial Y} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

$$X_4 = Y\frac{\partial}{\partial X} - X\frac{\partial}{\partial Y} - (\epsilon y + Y)\frac{\partial}{\partial x} + (\epsilon x + X)\frac{\partial}{\partial Y},$$

where $\epsilon = \ell/m$. $\mathfrak{iso}(M, g)(p)$ has at most rank 3 at every point p, so the dynamics are not quasilinearizable.

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