Analysis of orbits arising in piecewise-smooth discontinuous maps

Harish K Pillai
work done with
Bhooshan Rajpathak & Santanu Bandyopadhyay

Department of Electrical Engineering,
Indian Institute of Technology Bombay,
Mumbai, India-400076

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Why piecewise-smooth discontinuous maps?

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Examples include

- Controlled buck converter
- Boost converter in discontinuous mode
- Impact oscillators

1D linear piecewise smooth map

Linear piecewise smooth map

$$x_{n+1} = f(x_n, a, b, \mu, l) = \begin{cases} ax_n + \mu & for \quad x_n \leq 0 \\ bx_n + \mu + \ell & for \quad x_n > 0 \end{cases}$$

- The jump discontinuity is at $x = 0$
- a and b are the slopes of the affine maps on either side of discontinuity
- ℓ is the height of the “jump”
- μ is the parameter to be varied

Our interest

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- If yes, then can these orbits be characterized, classified $\cdots$

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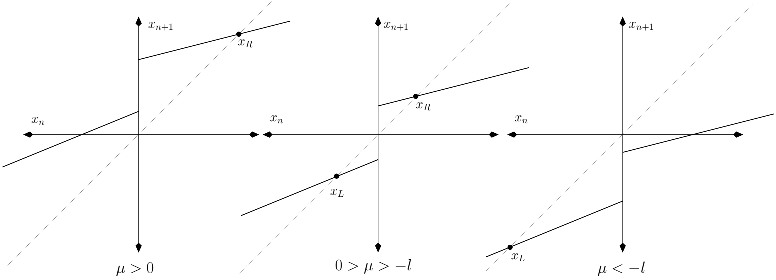

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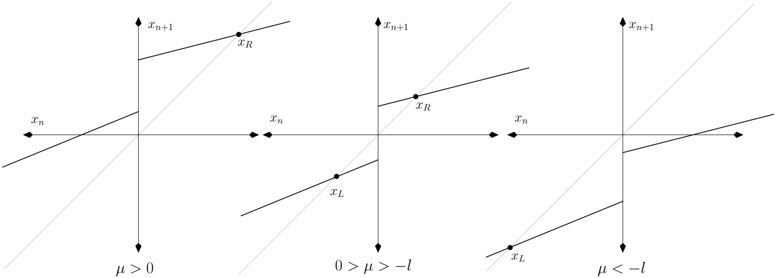

Equilibrium point in the left half $x_L = \frac{\mu}{1-a}$
Equilibrium point in the right half $x_R = \frac{\mu+\ell}{1-b}$

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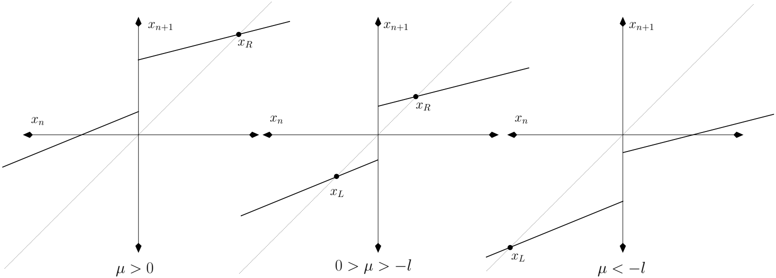

No chance of a periodic orbit !!

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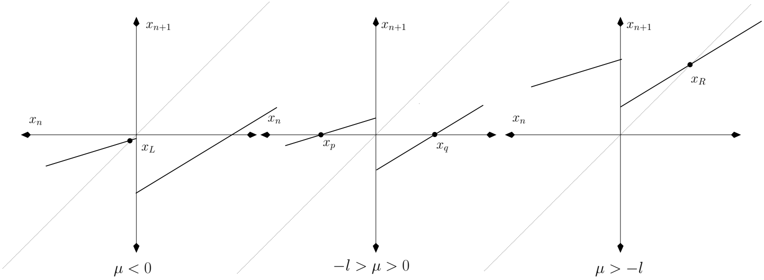

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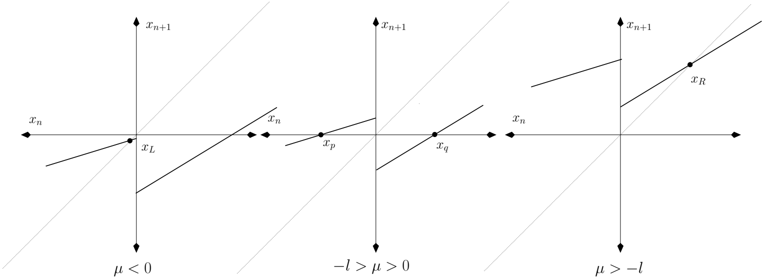

Orbits can exist if $0 < \mu < -\ell$
Set $\ell = -1$ and therefore $0 < \mu < 1$

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- Given a particular sequence of points $\{x_n\}_{n\geq 0}$ through which the system evolves, one can code this sequence into a sequence of $\mathscr{L}$s and $\mathscr{R}$s
- A periodic orbit has a string of $\mathscr{L}$s and $\mathscr{R}$s that keeps repeating. This repeating string is a pattern and denoted by σ

Some more definitions

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- $\mathscr{L}^n\mathscr{R}$ is a $\mathscr{L}$-prime pattern
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- A pattern made up of two or more prime patterns is a composite pattern

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For $a, b \in (0, 1)$, $\mathscr{L}$-prime and $\mathscr{R}$-prime patterns of any length are admissible

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Consider the pattern $\mathscr{L}^n\mathscr{R}$. The length of this pattern is $n+1$. From the map, one gets the following inequalities:

$$\begin{aligned} x_0 &\leq 0, \\ x_1 &= ax_0 + \mu \leq 0, \\ x_2 &= ax_1 + \mu \leq 0, \\ &= a^2x_0 + (a+1)\mu \leq 0, \\ &\vdots \\ x_{n-1} &= a^{n-1}x_0 + \mu S^a_{n-2} \leq 0, \\ x_n &= a^n x_0 + \mu S^a_{n-1} > 0, \\ x_{n+1} &= x_0 = a^n b x_0 + (bS^a_{n-1} + 1)\mu - 1 \leq 0. \end{aligned}$$

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$$P_{\mathscr{L}^{\mathrm{n}}\mathscr{R}} = \left(\frac{a^n}{S^a_n}, \quad \frac{a^{n-1}}{a^{n-1}b + S^a_{n-1}}\right]$$

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Showing that $P_{\mathscr{L}^{\mathrm{n}}\mathscr{R}} \neq \emptyset$ does the job

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- Can we characterize all the possible types of admissible patterns?
- For a given n, how many distinct patterns exist with period n?
- Is there an algorithm that generates only the possible admissible patterns of period n?

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- Similar limits can be found for runs of n symbols

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Lemma

For $a, b \in (0,1)$, all the admissible composite patterns are made up of either $\mathscr{L}$-prime patterns or $\mathscr{R}$-prime patterns but not both. Every composite pattern is a combination of exactly two prime patterns of successive lengths.

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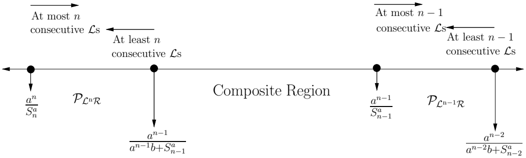

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- $\phi(18) = 6 - 1, 5, 7, 11, 13, 17$
- Thus there are patterns of length 18 with $1, 5, 7, 11, 13, 17$ $\mathscr{L}$s in them

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$\mathcal{RLRLLRLRLLRLLRLRLLRLL}$

101001010010010100100

↑ μ_2

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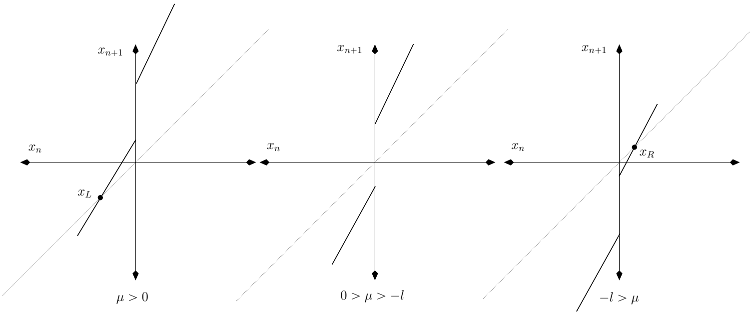

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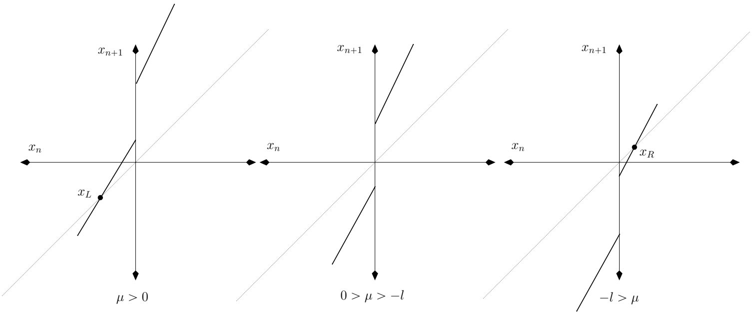

No chance of a periodic orbit !!

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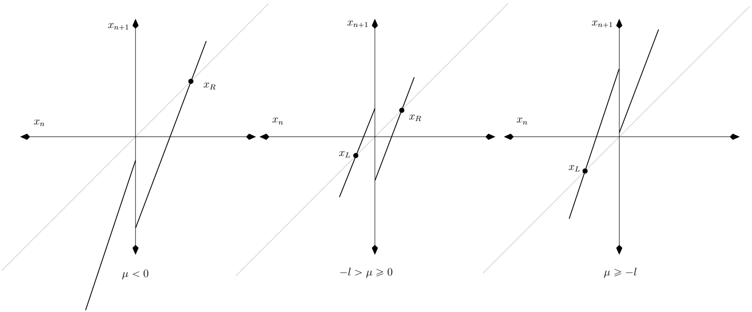

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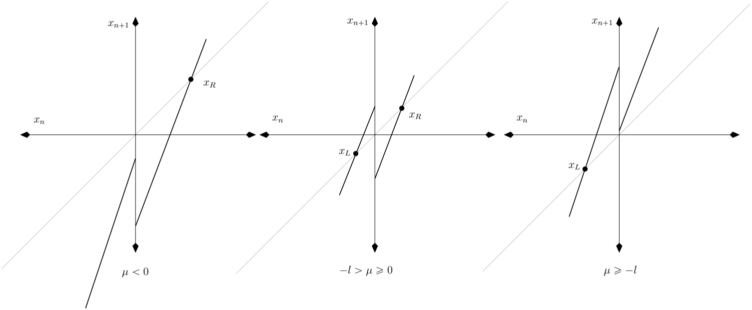

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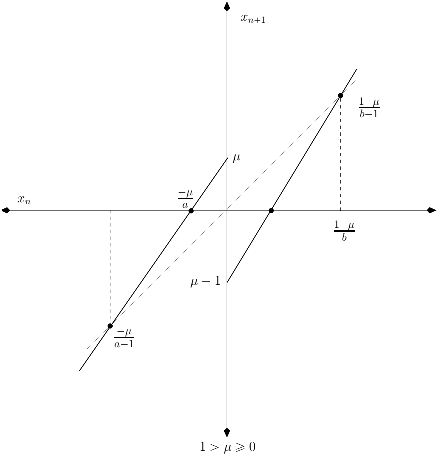
- Chaotic orbits exist !!

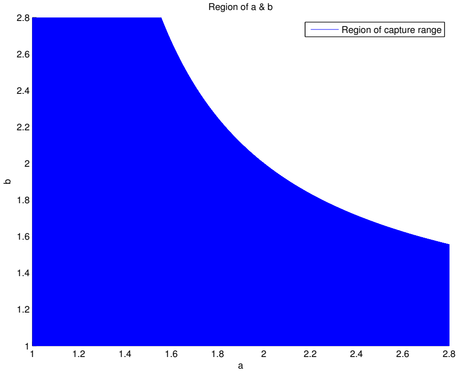
Chaotic orbits

Assumption: $a, b > 1$
Why?



Chaotic orbits

Assumption: $a, b > 1$
Capture range for μ is $\left(\frac{a-1}{a}, \frac{1}{b}\right)$



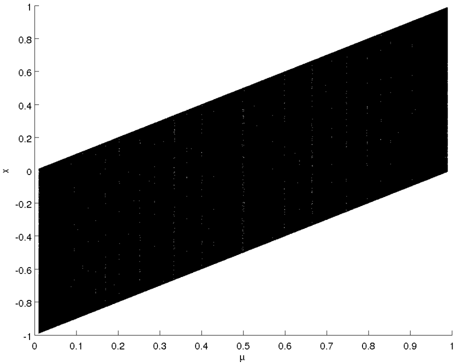
Only for values of a, b in blue – chaotic orbits
$1 < b < \frac{a}{a-1}$ and $1 < a < \frac{b}{b-1}$

Chaotic orbits

Assumption: $a, b > 1$

Some pictures

For $a = 1.01$, $b = 1.01$



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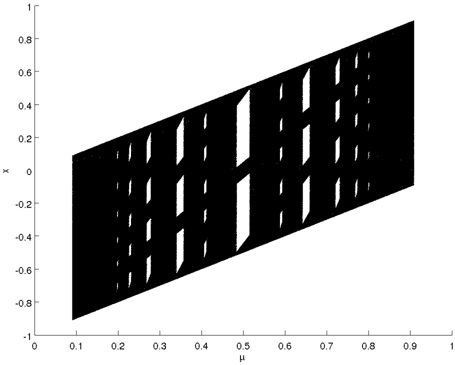
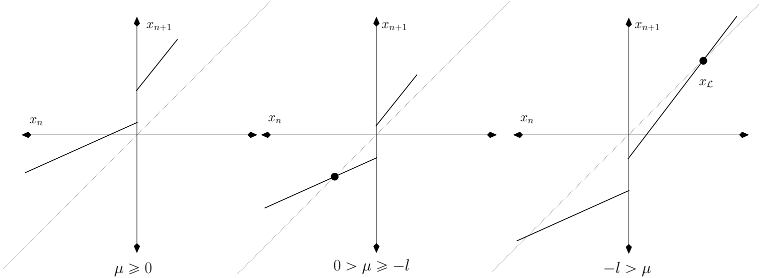
For $a = 1.1$, $b = 1.1$



Other cases – results

Assumption: $0 < a < 1$ and $b > 1$

For $\ell < 0$



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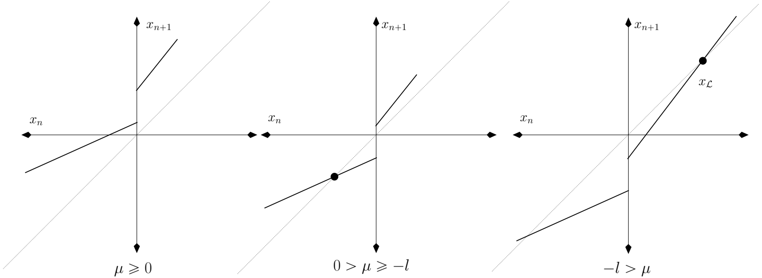

No orbits !!

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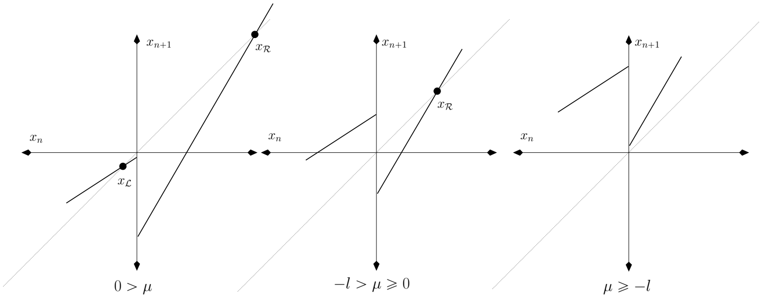

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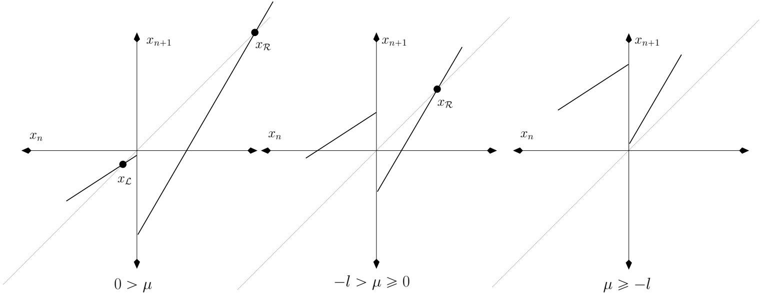

Orbits possible...

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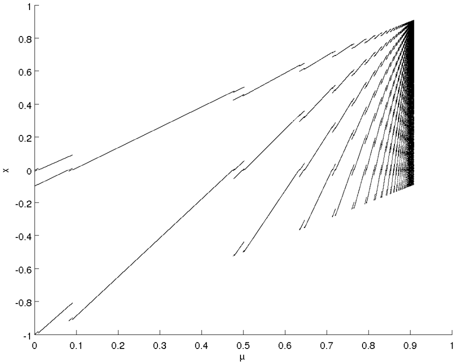
Some pictures

For $a = 0.1$ and $b = 1.1$

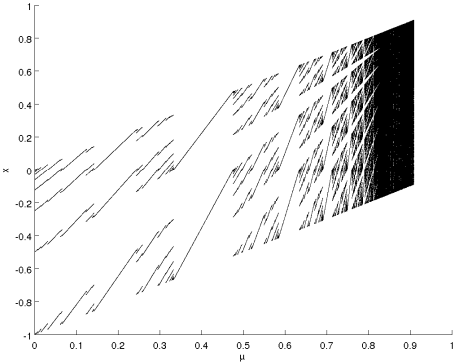


Other cases – results

Assumption: $0 < a < 1$ and $b > 1$

Some pictures

For $a = 0.5$ and $b = 1.1$

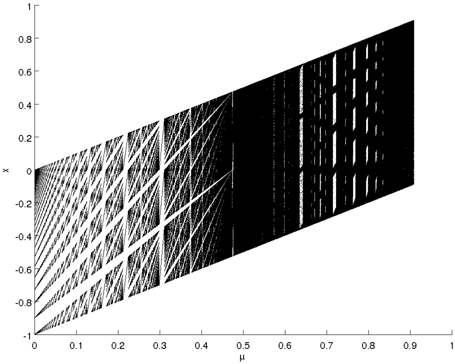


Other cases – results

Assumption: $0 < a < 1$ and $b > 1$

Some pictures

For $a = 0.9$ and $b = 1.1$

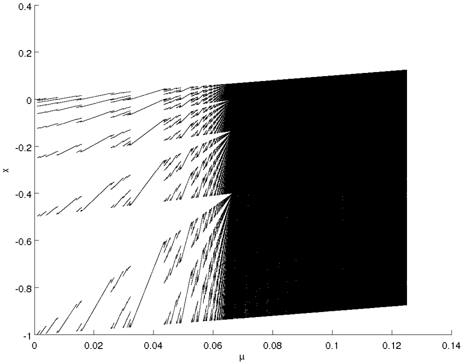


Other cases – results

Assumption: $0 < a < 1$ and $b > 1$

Some pictures

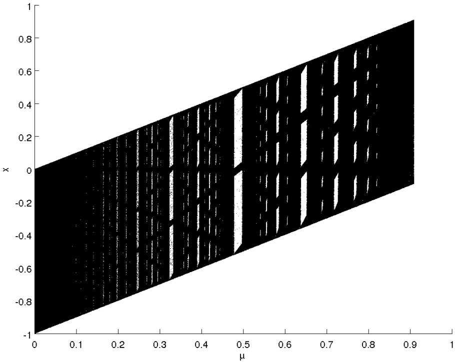
For $a = 0.5$ and $b = 8$



Boundary cases

By pictures

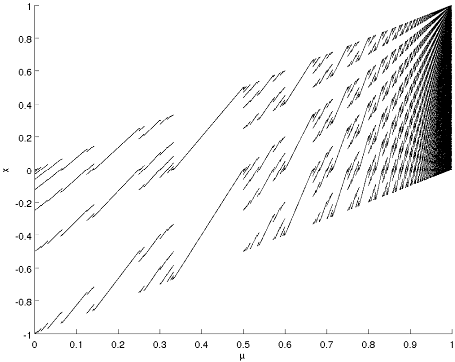
For $a = 1$ and $b = 1.1$



Boundary cases

Some pictures

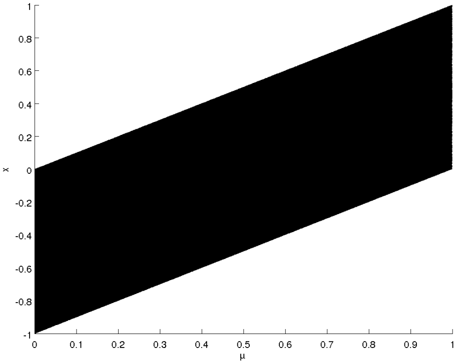
For $a = 0.5$ and $b = 1$



Boundary cases

Some pictures

For $a = 1$ and $b = 1$



Thank you very much