Synchronization and Nonlinear Integral Control

Indian Institute of Technology

Bombay

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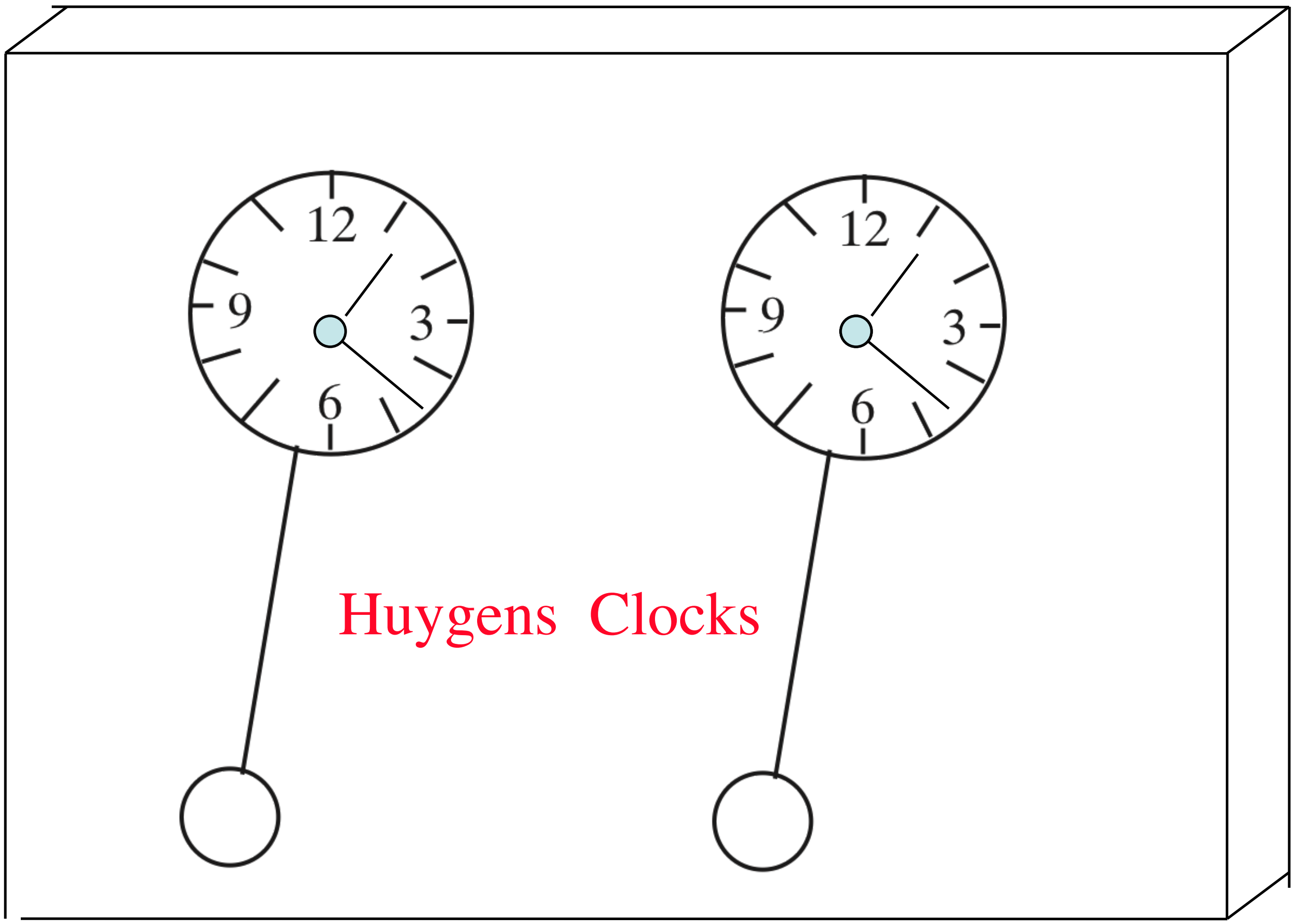
Roger Brockett

Harvard University

What is to be explained and what tools will help?



Today's communication networks require precise synchronization.

Possible First Question

Let $Q = Q^T > 0$ and consider

$$\ddot{x} + Qx = f(x, \dot{x})$$

what is the simplest, physically

plausable, choice of f

that results in synchronization?

Possible Second (Better) Question

Let $Q = Q^T > 0$ and consider

$$\ddot{x} + Qx = f(z) \; ; \; \dot{z} = g(x, z)$$

what is the simplest, physically

plausable, choice of f and g

that results in synchronization?

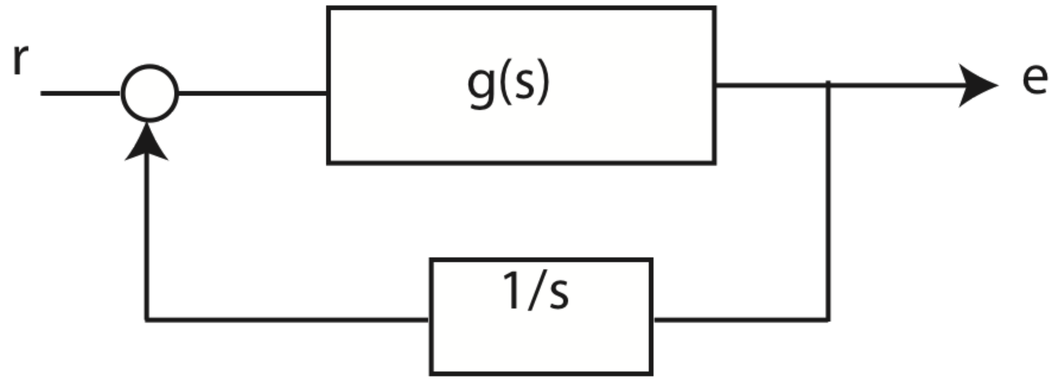
Properties of Huygens "Synchronization"

1. Independent of coupling strength (unmodeled)
2. Seems to equalize the originally unknown periods
3. Does not fix the relative amplitudes

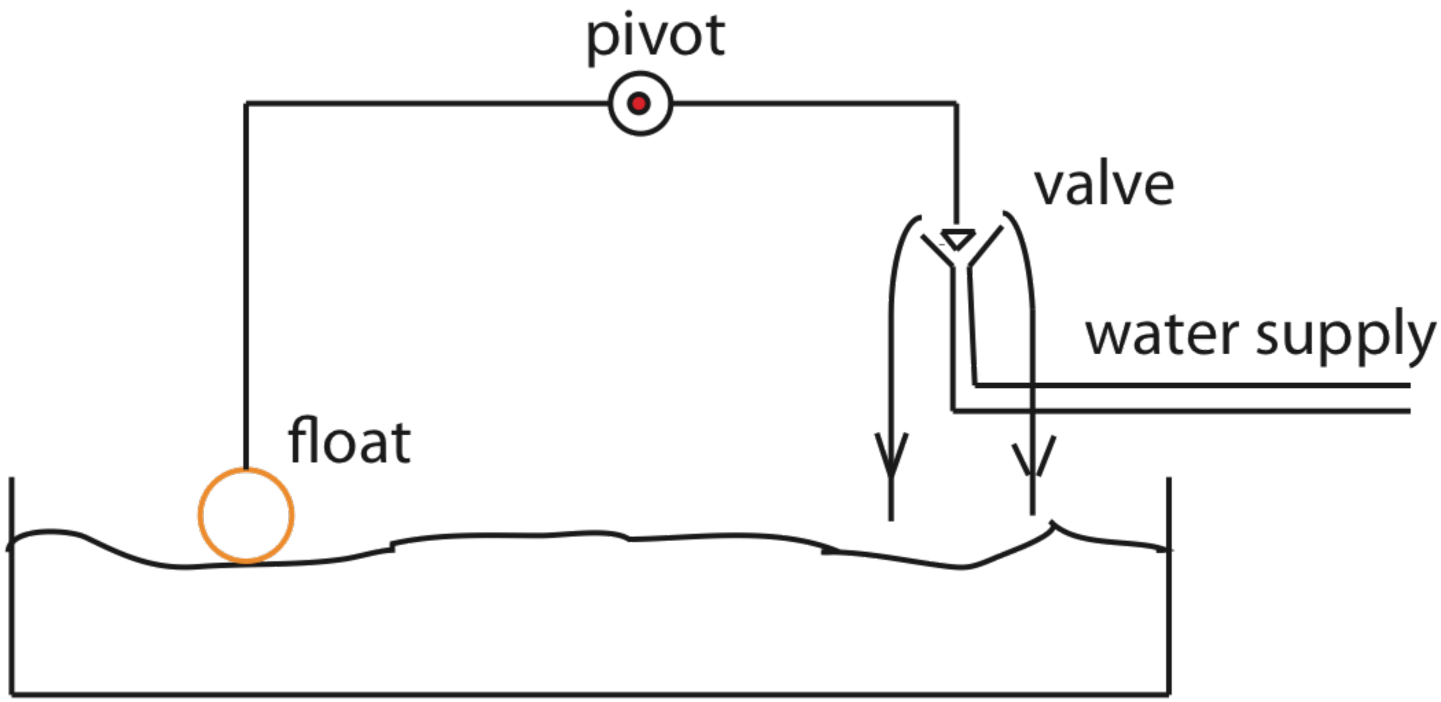
Uncertain Plant; Integral Control

1. With system unknown, it fixes the steady state value precisely.

2. Drives the error to zero but time constants can be large.



Uncertain Plant; Integral Control



classic example of integral control

$$\dot{h} = k \cdot (h_0 - h)$$

Problem Statement

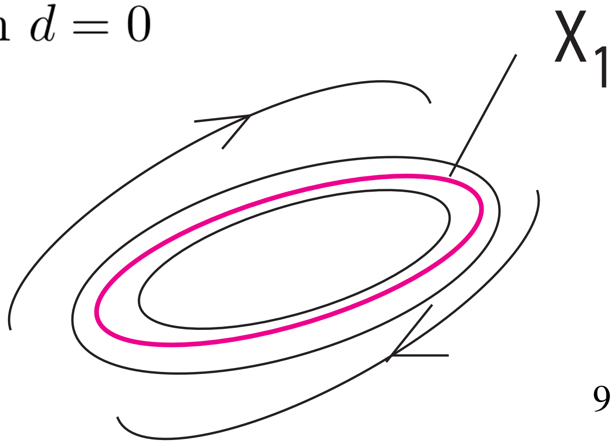
Let $X \subset \mathbb{R}^n$ be a asymptotically stable submanifold for $\dot{x} = f(x)$. By the *submanifold stabilization problem* for $\dot{x} = f(x) + \sum g_i(x)u_i$, we understand the problem of finding a control law $u(x)$ such that a given submanifold $X_1 \subset X$ is attracting.

There are many such stabilization questions that find use in control some quite interesting from a mathematical point of view.

A Stabilization Theorem: Background

Consider
$\dot{x} = f(x) + \sum g_i(x)u_i$; $x \in \mathbb{R}^n$ Let $X \subset \mathbb{R}^n$ be a compact, invariant submanifold for $\dot{x} = f(x)$ and assume that X is asymptotically stable. Let $X_1 \subset X$ also be invariant under the flow defined by $\dot{x} = f(x)$. Then if $\{g_i\}$ span the normal bundle of X_1 in a tubular neighborhood of X_1 then there exists a control law $u = u(x)$ that makes X_1 asymptotically stable.

Proof: Because X is assumed to be asymptotically stable we can limit our attention to initial conditions in X. For x in a neighborhood of X_1 let $d(x)$ denote the euclidean distance to X_1. Pick the u_i so that $\langle \nabla d, g_i u_i \rangle \leq 0$ Because collectively, the g_i span X_1, we see that along trajectories the distance is monotone decreasing and vanishes only when $d = 0$



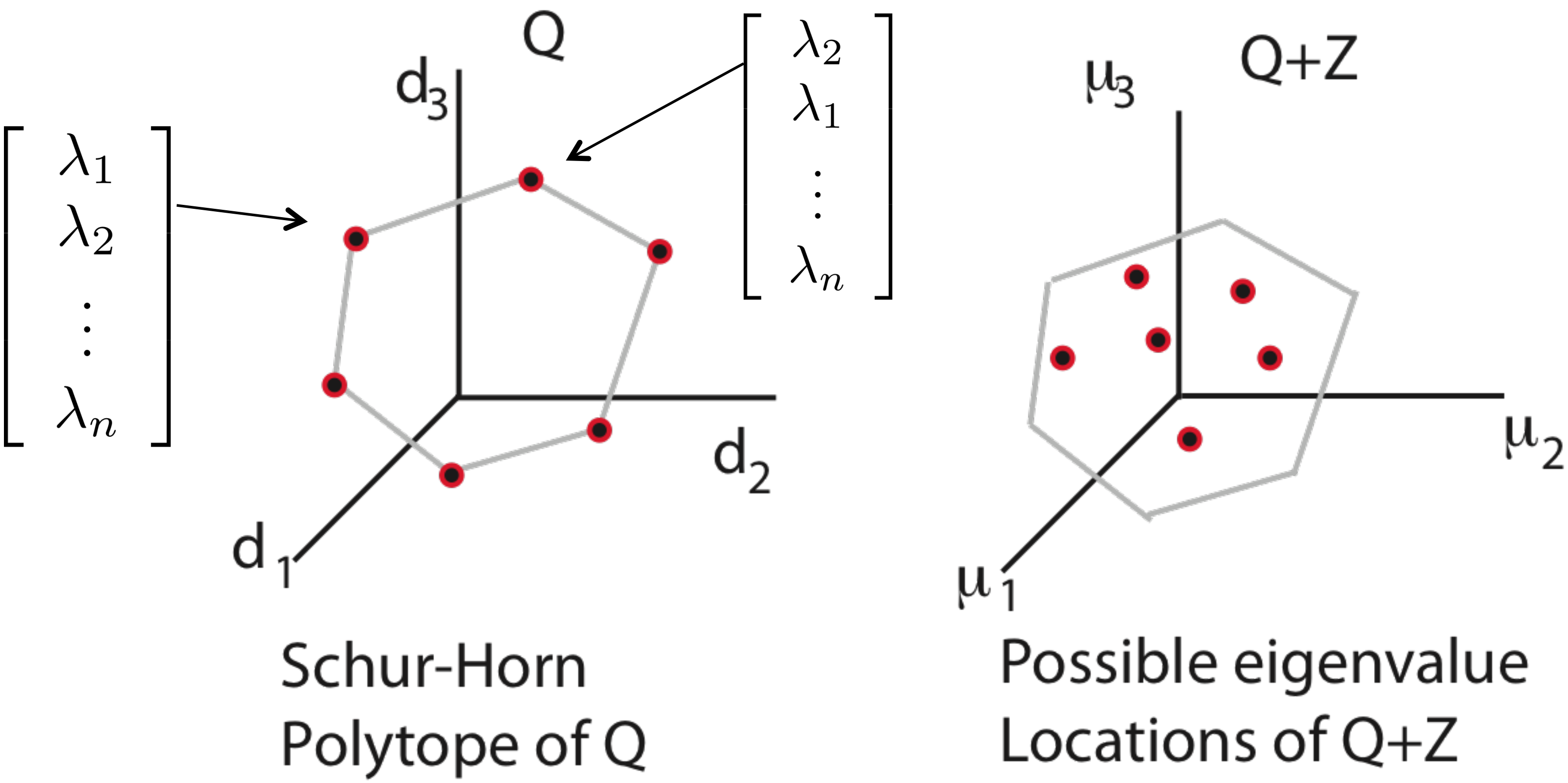
Background: The Schur-Horn Theorem

Consider the real symmetric matrix Q with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} \text{ and } \begin{bmatrix} \lambda_2 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \text{ and n! } - 2 \text{ more } \cdots$$

The Schur-Horn theorem says the possible diagonals of Q are just the convex combinations of these vectors.

In Pictures: Eigenvalues to Diagonals to Eigenvalues



In particular, by adding a skew-symmetric matrix we can make all eigenvalues real and equal.

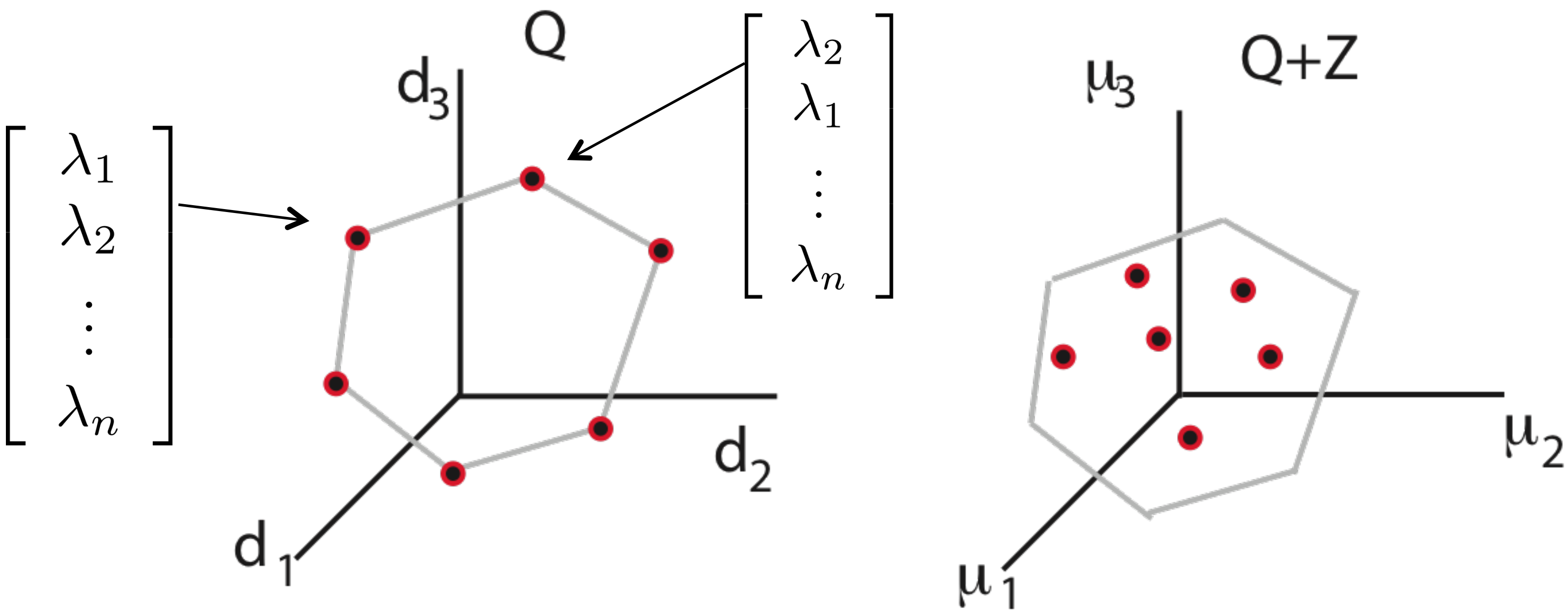
What is involved in proving that any such diagonal can be realized?

One approach to the proof is to look for the orthogonal matrix Θ such that $||\Theta^T Q\Theta - D||$ is minimized and then show that the minimum is zero under the Schur-Horn conditions. Because there are many local minima, arriving at a decisive conclusion requires a somewhat tedious second derivative calculation.

A New Variation on the Schur-Horn Theorem (An eigenvalue placement with multiplicity)

Given $Q = Q^T$ with eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_n\}$, and given any vector $[\mu_1, u_2, ..., \mu_n]$ in the Schur-Horn polytope defined by the eigenvalues of Q, there exists $Z = -Z^T$ such that the eigenvalues of $Q + Z$ are $[\mu_1, u_2, ..., \mu_n]$. Moreover, if the eigenvalues of $Q + Z$ are real, they must lie in this polytope and if they lie on the boundary of the polytope between two distinct eigenvalues of Q they must be associated with an elementary divisor of degree two or higher.

In Pictures: Eigenvalues to Diagonals to Eigenvalues



Schur-Horn Polytope of Q

Possible eigenvalue Locations of Q+Z

In particular, by adding a skew-symmetric matrix we can make all eigenvalues real and equal.

However if the eigenvalues are to be equal…..

Working out the Jordan normal form when $Q + \Omega$ is two-by-two with repeated eigenvalues one sees that that either $\Omega = 0$ or $Q + \Omega$ has a one-chain. The general result follows.

How to prove this?

By Schur-Horn we can find Θ such that the diagonals of $\Theta^T Q \Theta$ are the desired eigenvalues. To the matrix $\Theta^T Q \Theta$ add a skew-symmetric matrix Z such that the sum is either upper or lower triangular. Then revert to the original coordinate system to get $Q + \Theta Z \Theta$ which has the desired eigenvalues. We are not claiming anything about the geometric multiplicity of the eigenvalues of $Q + \Omega$ which will be greater than one in some cases.

An important refinement

Because eigenvalue placement involves n parameters if Q is n-by-n one can consider restricting Ω. Of interest in this context is minimizing its rank. Can one place eigenvalues with Ω of rank 2?

How are we to use this fact?

Consider an n-dimensional second linear system of the form
$\ddot{x} + Qx = 0$ with $Q = Q^T > 0$
The frequencies are the square roots of the eigenvalues of Q. We would like to find the simplest way to alter this equation in such a way as to obtain synchronization.
Simplest means lowest order terms in Taylor series.
We have observed that $Q + Z$ can be made to have equal, real, eigenvalues. Thus we look at $\ddot{x} + (Q + Z)x = 0$

What type of coupling can work?

Towards nonlinear integral control: what second order terms are available?

We do not want to make unnecessary assumptions on the clock mechanism. For example, if we model the escapement mechanism as a dynamical system the system is at least fourth order, etc. However, we will stick to linear models. Thus, we want to identify the full set of linearly independent quadratic integrals for such models

Characterizing the resources in terms of the number of systems (clocks) and their degrees of freedom.

This is actually a longer story. The problem can be reduced to a discussion of quadratic forms in m variables and their derivatives with respect to time For example $q(x, \dot{x}, y, \dot{y}, \ddot{y})$ In this case certain integrals such as $\dot{x}y + \dot{y}x$ are "exact" where others such as $\dot{x}y - \dot{y}x$ are not.

The exact terms do not offer interesting possibilities beyond those already present in x and dx/dt.

Controllability with Linear and Quadratic Drift

We associate with the system
$\dot{x} = Ax + Bu \; ; \; \dot{w}_i = x^T D_i x \; ; \; i = 1, 2, \cdots p$
a vector-matrix system
$\dot{x} = Ax + Bu \; ; \; \dot{W} = xx^T$
which we call the *covering system*. The name is suggested by the fact that the evolution of w can be viewed as a projection whereby W is mapped to w with $w_i = \langle D_i, W \rangle$.

Controllability with Linear and Quadratic Drift

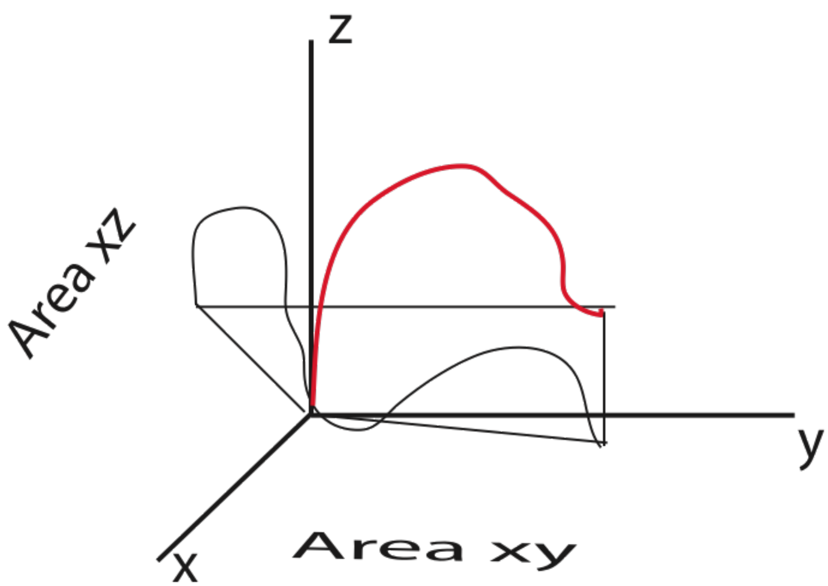
Let $\mathcal{D}_{AB}$ be the linear span of the reachable quadratic terms and let (A, B) be a controllable pair with $B^T B$ nonsingular. Then $\mathcal{D}_{AB}$ has dimension $nm - m(m-1)/2$,

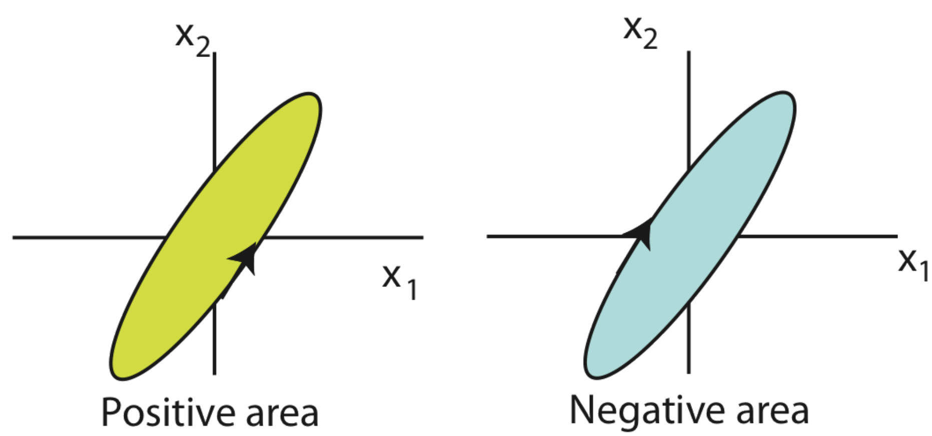
Of these, n are monotone increasing where n is the dimension of A. Thus if $n = 2$ and $m = 2$ there are three that are "phase sensitive" and are not so constrained.

Corrective Signal; the Lissajous figure in n-dimensions

The $n(n-1)/2$ projected areas

$$Z = \frac{1}{2}\int_0^T (x\dot{x}^T - \dot{x}x^T)\,dt$$





When x is 2-dimensional and sinusoidal

Corrective signal; relative phase

Suppose

$$x(t) = \begin{bmatrix} \sin t \\ \sin(t+\phi) \end{bmatrix} \; ; \; \dot{x} = \begin{bmatrix} \cos t \\ \cos(t+\phi) \end{bmatrix}$$

so that $x\dot{x}^T - \dot{x}x^T$ is determined by

$$\sin t \cos(t+\phi) = \sin t(\cos t \cos\phi + \sin t \sin(\phi))$$

whose average value is $(\sin\phi)/2$, thus providing an error signal for nulling the phase difference. Note that if $\phi(t) = (\omega_2 - \omega_1)t + \theta$ then this is $\sin((\Delta_\omega t + \theta)$

Potential models for frequency equalization

Direct Adjustment

$\ddot{x} + \epsilon f(\dot{x}) + x + \epsilon^2 (Q + x\dot{x}^T - \dot{x}x^T) = 0$

Not Huygens-like
No integration

Integral Action

$\ddot{x} + \epsilon f(\dot{x}) + x + \epsilon (Q + Z)x = 0$

$\dot{Z} = x\dot{x}^T - \dot{x}x^T$

Strength of interaction
not important

Numerical studies suggest that for small ϵ there is synchronization and mode locking but with a small irregular motion which averages out; the phase difference between the various oscillators is determined by Q but subject to a small jitter. One of our main points is that there is no solution to the averaging equations ordinarily used to establish periodic solutions.

A Numerical Example

An example

$$\ddot{x}_1 + .2(x_2^2 + \dot{x}_1^2 - 1)\dot{x}_1 + 1.1x_1 - x_3x_2 = 0$$

$$\ddot{x}_2 + .2(x_2^2 + \dot{x}_2^2 - 1)\dot{x}_2 + .9x_2 + x_3x_1 = 0$$

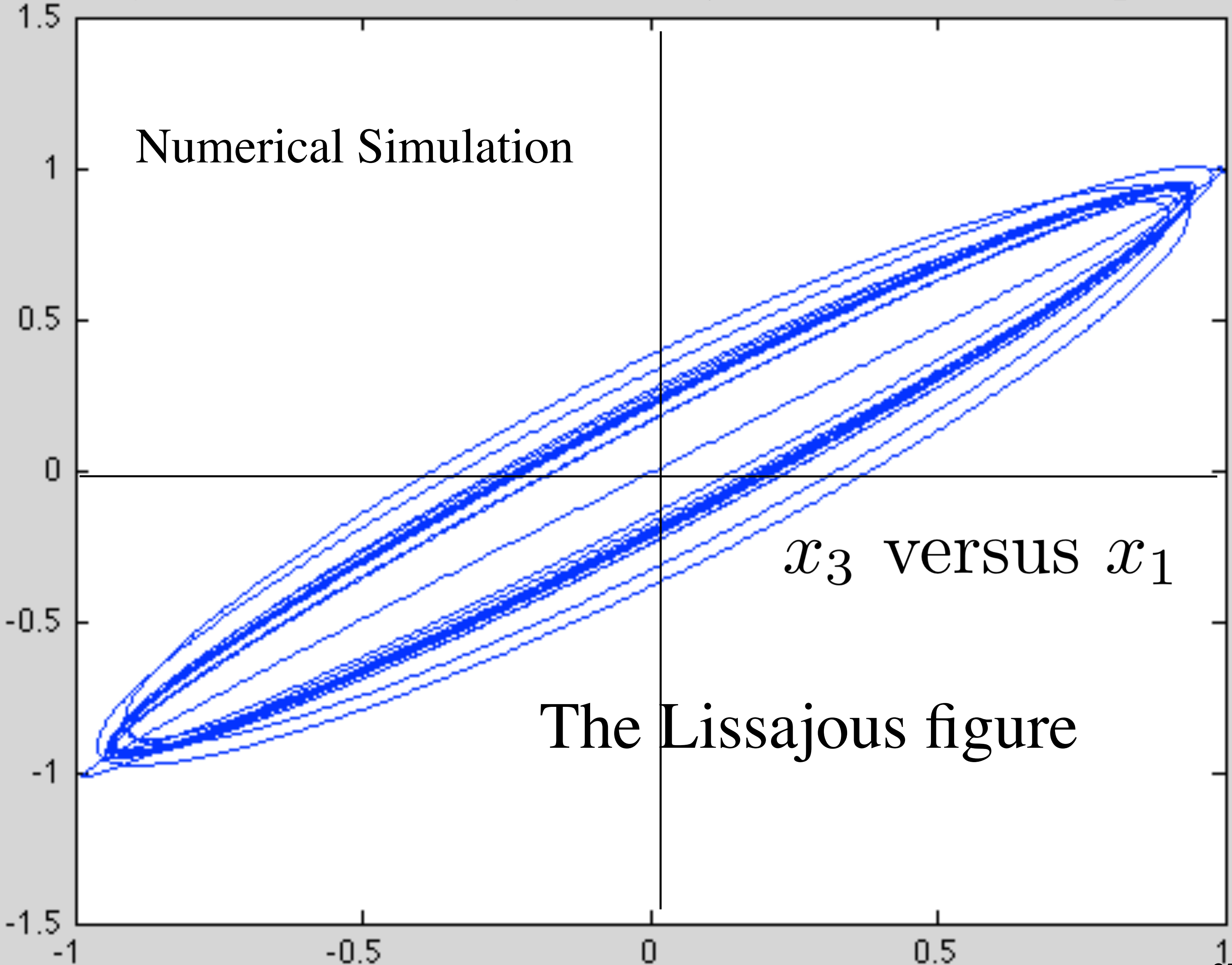
$\ddot{x}_3 + 100\dot{x}_3 + 10x_3 = 5(x_1\dot{x}_2 - x_2\dot{x}_1)$ overdamped DC gain $=1/2$

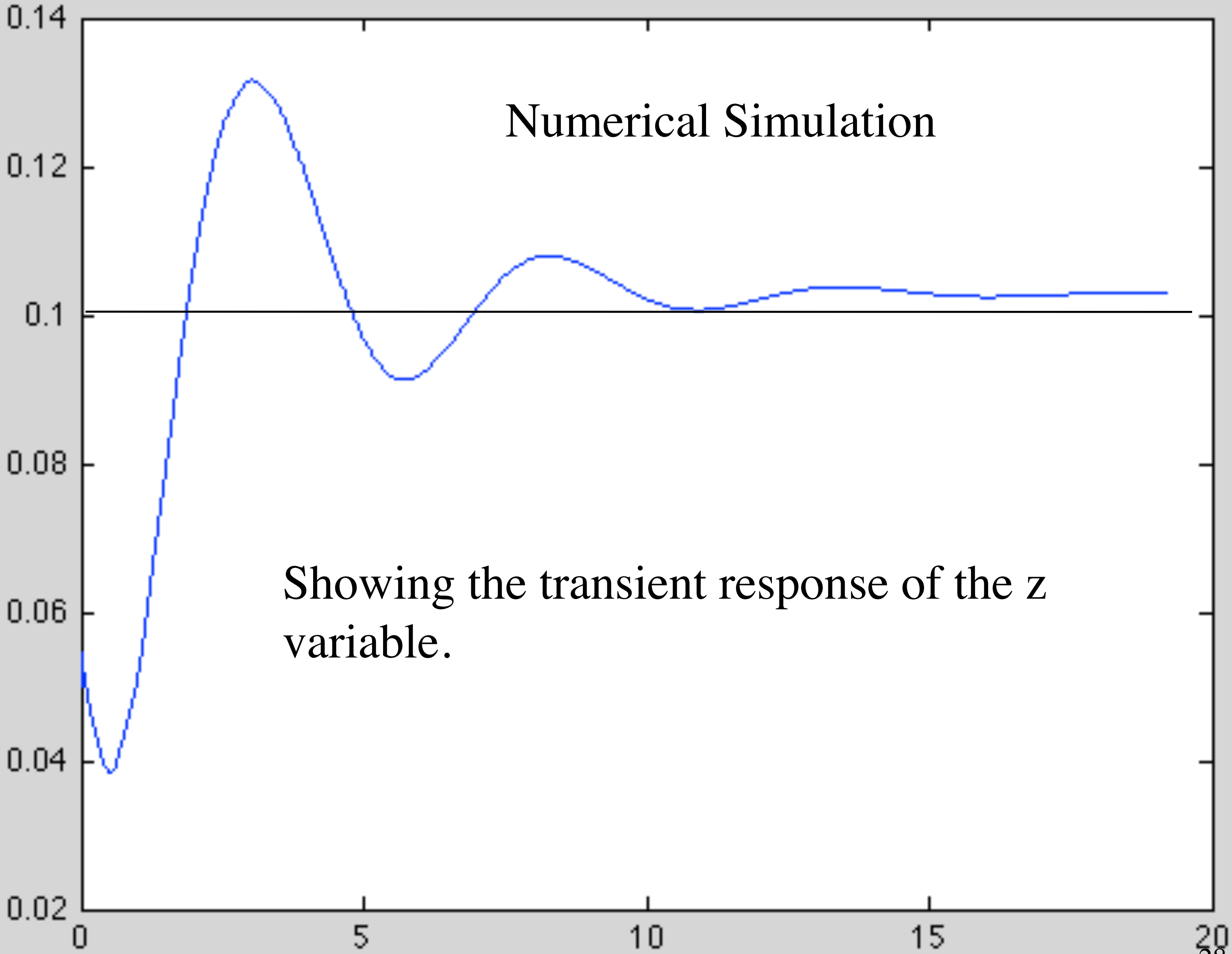
Observe that for $x_3 = .1$ the eigenvalues of the matrix

$$Q = \begin{bmatrix} 1.1 & -a \\ a & .9 \end{bmatrix}$$

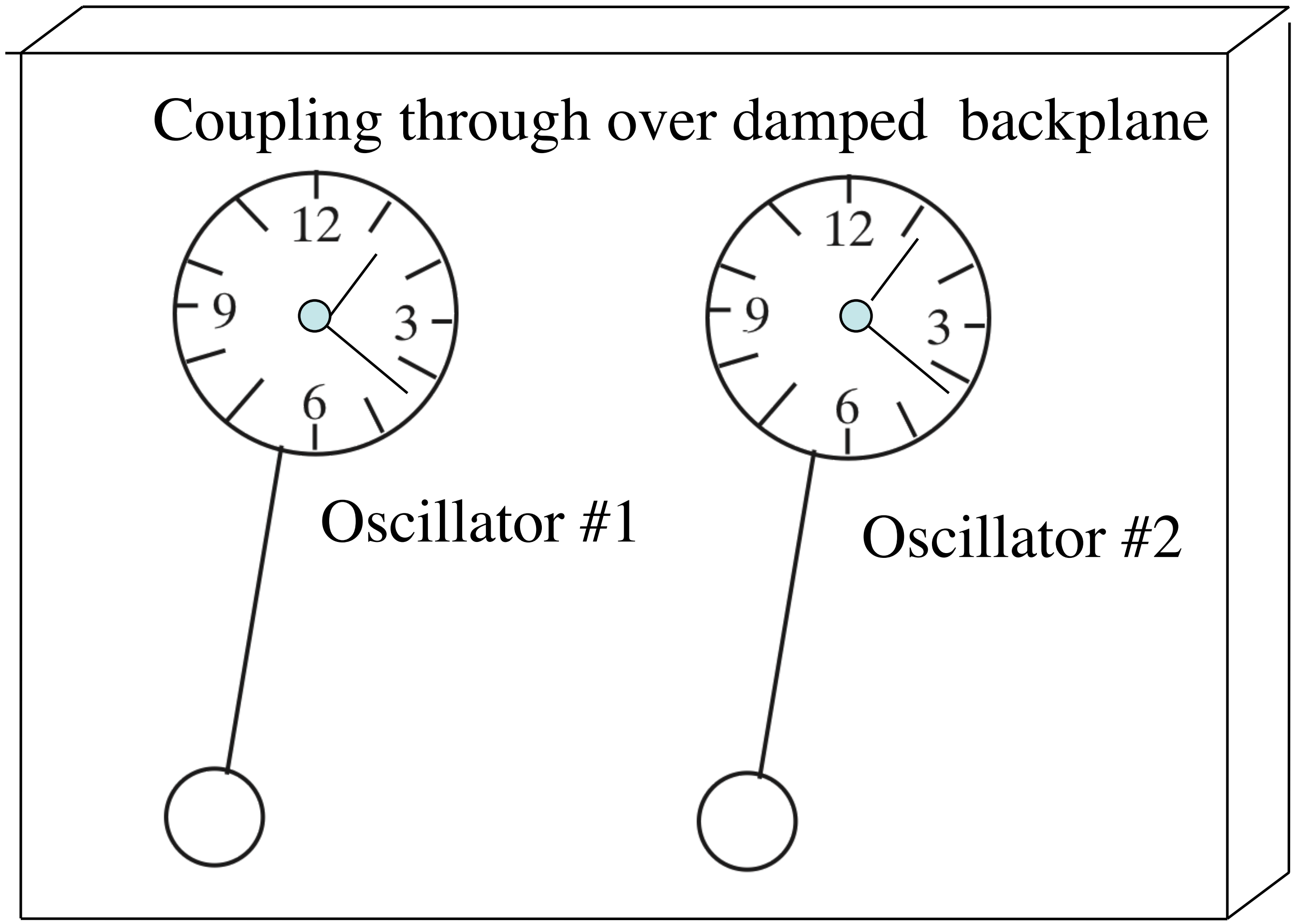
are are both 1.

Showing numerical convergence to synchronization with phase offset




Numerical Simulation
Showing the transient response of the z variable.
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
5
10
15
20

Undamped oscillators coupled by an over damped back plane. Nonlinearities “rectify” out-of-phase oscillations producing a corrective signal.



Is the solution truly periodic?

Let Q be a symmetric detuning matrix with zero trace.
$\ddot{x} + \epsilon D(x, \dot{x})\dot{x} + x + \epsilon(Q + Z)x = 0$
$\dot{Z} = -\alpha Z + x\dot{x}^T - \dot{x}x^T$
The averaging equations relating to a possible periodic solution, demand that
$x(t) = a \sin \omega t + b \cos \omega t \ ; \ Z = Z_0 \ ; \ a^T a + b^T b = 1$, etc.
The column vectors a and b, must satisfy
$(Q + Z)a = 0 \ ; \ (Q + Z)b = 0 \ ; \ Z_0 = 2(ab^T - ba^T)$
These equations have no real solutions because the null space of $Q + Z$ is one dimensional so that $a = \gamma b$

To Disarm Potential Defenders of Huygens (1629 – 1695)

It is true that many other, quite different, coupling laws may be in effect here. However, it is part of the story that the coupling is weak and weak means low order terms in the Taylor series expansion. No linear coupling will work. Our terms are second order but, as we have presented it, not the most general second order term. However the most general second order terms consist of ours plus terms that are dominated by the first order terms given.

For reference: Newton 1642 – 1727

For further details, see

Roger Brockett, (2013) “Synchronization without Periodicity”,
Festschrift in Honor of Uwe Helmke
http://users.cecs.anu.edu.au/~trumpf/UH60Festschrift.pdf

Roger Brockett, (2013) “ Controllability with quadratic drift”,
MATHEMATICAL CONTROL AND RELATED FIELDS
Volume 3, Number 4, December 2013

Abstract Version of the theorem on eigenvalue adjustment

A Given a $\mathbb{Z}_2$ graded Lie algebra L we have the L_0 sub algebra and the corresponding Lie group G_0. Consider the Kostant version of Schur-Horn and the possible spectra of $L_1 + L_0$.

Conclusions

We have given a result on eigenvalue placement for symmetric matrices perturbed by skew-symmetric matrices.

We have given an argument that Huygens synchronization involves a type of integral control, albeit a nonlinear form.

We have described how the “first bracket” controllable integrals can provide the necessary integral control.

We suggest that because averaging theory shows there is no periodic solution for the obvious model, whereas numerical simulation shows apparent synchronization, in fact, Huygens synchronization is not actually synchronization but highly confined near periodic irregular motion.