Spacecraft Attitude Control using CMGs: Singularities and Global Controllability

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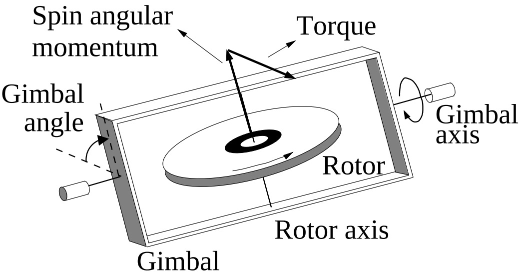
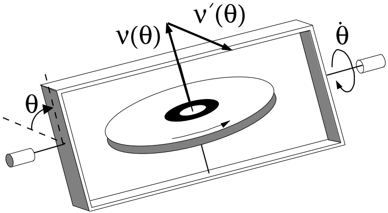


- Reaction wheel: gimbal stationary, rotor speed varies
- CMG: Rotor speed constant, gimbal moves
- Variable speed CMG: Rotor speed varies, gimbal moves

Single-gimbal CMG

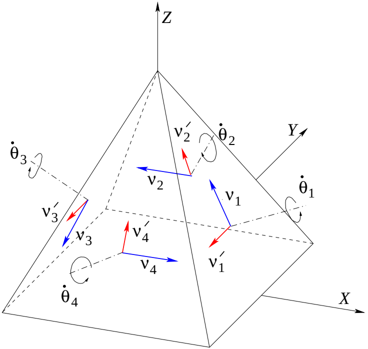


- Assumption: Gimbal inertia negligible, gimbal rate small
- Consequence: CMG spin angular momentum directed along the rotor axis, fixed in magnitude, function only of gimbal angle

CMG angular momentum $\nu(\theta)$, actuation torque $-\nu'(\theta)\dot{\theta}$

- For control authority as well as redundancy, multiple CMGs are used in a collection called an *array*

A pyramid array



Schematic of a pyramid array

CMG Arrays: Notation and Terminology

- Consider a CMG array comprising q single-gimbal CMGs
  - CMG configuration $\theta = [\theta_1, \ldots, \theta_q]^{\mathrm{T}} \in \mathbb{T}^q$
  - Spin angular momentum of ith CMG $\nu_i : \mathrm{S}^1 \to \mathbb{R}^3$
  - Total spin angular momentum $\nu : \mathbb{T}^q \to \mathbb{R}^3$ given by

$$\nu(\theta) \stackrel{\text{def}}{=} \nu_1(\theta_1) + \cdots + \nu_q(\theta_q)$$

  - Actuation torque $= -\frac{\partial \nu}{\partial \theta}(\theta)\dot{\theta}$
    - Jacobian

$$\frac{\partial \nu}{\partial \theta}(\theta) = \left[\nu_1'(\theta_1), \ldots, \nu_q'(\theta_q)\right]_{3\times q}.$$

  - Momentum volume $\mathcal{V} = \nu(\mathbb{T}^q) \subset \mathbb{R}^3$
  - Momentum envelope $=$ topological boundary of $\mathcal{V}$

Momentum Envelope of a Pyramid Array

Taken from G. Margulies and J. N. Aubrun, “Geometric theory of single-gimbal control moment gyro systems,” *Journal of the Astronautical Sciences*, vol. XXVI, 1978

- Body components of total (spacecraft + CMG) angular momentum

$$H = J\omega + \nu(\theta)$$

J	$=$	3×3 moment-of-inertia matrix
ω	$=$	body components of angular velocity of body frame relative to an inertial reference frame

- Euler's equations assuming no external torque

$$\dot{H} + \omega \times H = 0$$

- Attitude dynamical equation

$$J\dot{\omega} = -\omega \times [J\omega + \nu(\theta)] \quad \underbrace{-\frac{\partial \nu}{\partial \theta}(\theta)\dot{\theta}}_{\text{actuation torque}}$$

Attitude Control Using CMGs: Prevalent Approach

- First, find the actuation torque profile $\tau(\cdot)$ required to achieve the desired spacecraft behavior
- Next, solve

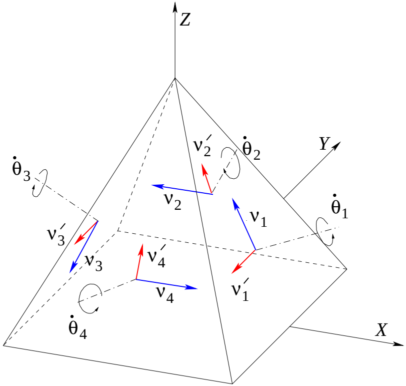
$$\tau(t) = -\frac{\partial \nu}{\partial \theta}(\theta(t))\dot{\theta}(t)$$

to find the gimbal rate profile $\dot{\theta}(\cdot)$ that yields the required actuation torque profile $\tau(\cdot)$

- Views the CMG array as only a torque-producing device
- Can we always do this?

- A CMG configuration $\theta \in \mathbb{T}^q$ is a *singular configuration* if

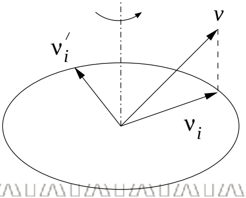
$$\text{rank } \frac{\partial \nu}{\partial \theta}(\theta) < 3$$



- A unit vector $v \in S^2$ is a *singular direction* corresponding to a singular configuration $\theta \in \mathbb{T}^q$ if

$$v^{\mathrm{T}} \frac{\partial \nu}{\partial \theta}(\theta) = 0, \text{ that is, } v^{\mathrm{T}} \nu_i'(\theta_i) = 0 \; \forall\, i$$

  - All actuation torques possible at θ are confined to $\{v\}^{\perp}$
- Every singular configuration posseses a singular direction
- Every $v \in S^2$ is a singular direction for some θ



What Happens to the Prevalent Approach?

- Encounters difficulties at or near singular configurations
- Has led to
  - Detailed studies of geometric properties of singular configurations
  - Large body of work on steering algorithms for generating gimbal rate profiles that yield required torque profiles without running into singular configurations
- Steering algorithms
  - Are partly based on heuristics
  - Have been successful in practice
  - Lack theoretical guarantees

Unanswered Questions

Is it important to avoid singularities?

- Maybe, if arbitrary torque profiles need to be generated
  - For example, attitude trajectory tracking
- Maybe not
  - For example, asymptotic attitude stabilization, state-to-state steering

Is the local underactuation caused by singularities really a problem?

- Underactuation does not always present an obstacle to stabilization, controllability

Exactly which system-theoretic properties do singular configurations pose an obstruction for?

Consider the combined dynamics of the spacecraft and the CMG array

- Treat gimbal rates as inputs
- Apply control-theoretic tools to determine
  - Global controllability (subject of this talk)
    - S. P. Bhat and P. K. Tiwari, "Controllability of spacecraft attitude using control moment gyroscopes," *IEEE TAC*, Vol. 54, March 2009.
  - Local controllability and stabilizability (subject of the next talk)
    - S. P. Bhat and A. A. Paranjape, "Small-time local controllability and stabilizability of spacecraft attitude dynamics under CMG actuation," *SIAM Journal of Control and Optimization*, Vol. 52, March 2014.

Attitude Representation

- Attitude represented by $R \in \mathrm{SO}(3) \stackrel{\text{def}}{=} \{S \in \mathbb{R}^{3\times 3} : S^{\mathrm{T}}S = I, \det S = 1\}$ such that
  $R \times$ body components $=$ inertial components
- $\mathrm{SO}(3)$ is a Lie group with Lie algebra

$$\mathrm{so}(3) = \{G \in \mathbb{R}^{3\times 3} : G = -G^{\mathrm{T}}\}$$

- The usual cross product on $\mathbb{R}^3$ gives rise to a Lie algebra isomorphism $(\cdot)^{\times} : \mathbb{R}^3 \to \mathrm{so}(3)$ given by

$$a^{\times} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \; a \in \mathbb{R}^3$$

$$\dot{R} = R\omega^{\times}$$

Conservation of Angular Momentum

Inertial components of the total angular momentum are constant

$$R[J\omega(t) + \nu(\theta(t))] = \mu \stackrel{\text{def}}{=} R[J\omega(0) + \nu(\theta(0))]$$

- Dynamics not controllable on $\mathrm{TSO}(3) \times \mathbb{T}^q$
- Only need to consider dynamics on an angular momentum level set
  - Level set is diffeomorphic to $\mathrm{SO}(3) \times \mathbb{R}^3$ for each $\mu \in \mathbb{R}^3$

$$\omega = J^{-1}[R^{\mathrm{T}}\mu - \nu(\theta)]$$

$$\begin{aligned}\dot{R} &= R[J^{-1}(R^{\mathrm{T}}\mu - \nu(\theta))]^{\times}\\ \dot{\theta} &= u\end{aligned}$$

- Defines a family of input-affine control systems parametrized by $\mu \in \mathbb{R}^3$ with
  - State $(R, \theta) \in \mathrm{SO}(3) \times \mathbb{T}^q$
  - Input $=$ gimbal rate vector $u \in \mathbb{R}^q$
  - Drift vector field $f_\mu(R, \theta) = (R[J^{-1}(R^{\mathrm{T}}\mu - \nu(\theta))]^{\times}, 0)$
  - Control vector field $g_i(R, \theta) = (0, e_i)$
- Fixing μ is same as restricting to an angular momentum level set
- No reduction applied so far

TATA

$\mathcal{R}(x,t)$ = set of states reached at time t by starting from $x \in \mathrm{SO}(3) \times \mathbb{T}^q$ at time 0

$\cup_{t\geq 0}\mathcal{R}(x,t)$ = set of states that can be reached in finite time by starting from $x \in \mathrm{SO}(3) \times \mathbb{T}^q$ at time 0

Dynamics are

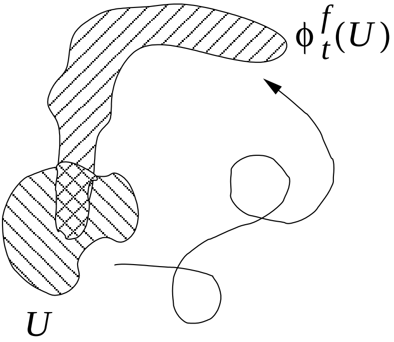
- *strongly accessible* if $\mathcal{R}(x,t)$ has nonempty interior in $\mathrm{SO}(3) \times \mathbb{T}^q$ for every x and $t > 0$
- *accessible* if $\cup_{t\geq 0}\mathcal{R}(x,t)$ has nonempty interior in $\mathrm{SO}(3) \times \mathbb{T}^q$ for every x
- *controllable* if $\cup_{t\geq 0}\mathcal{R}(x,t) = \mathrm{SO}(3) \times \mathbb{T}^q$

$$
\begin{aligned}
\xi_1(R,\theta) \overset{\text{def}}{=} [g_1, f_\mu](R,\theta) &= (-R(J^{-1}\nu_1')^\times, 0) \\
\xi_2(R,\theta) \overset{\text{def}}{=} [g_1, \xi_1](R,\theta) &= (-R(J^{-1}\nu_1)^\times, 0) \\
\xi_3(R,\theta) \overset{\text{def}}{=} [\xi_1, \xi_2](R,\theta) &= (-R(J^{-1}\nu_1 \times J^{-1}\nu_1')^\times, 0)
\end{aligned}
$$

- ν_1, ν_1' are linearly indpendent at every θ
- $J^{-1}\nu_1$, $J^{-1}\nu_1'$ and $J^{-1}\nu_1 \times J^{-1}\nu_1'$ are linearly independent at every θ
- The vector fields $\xi_1, \xi_2, \xi_3, g_1, \ldots, g_q$ span the tangent space to $\mathrm{SO}(3) \times \mathbb{T}^q$ at every (R,θ)

The dynamics are strongly accessible and accessible on $\mathrm{SO}(3) \times \mathbb{R}^3$ for every choice of μ

The vector field f is WPPS if, for every open set U and every $T > 0$, there exists $t > T$ such that $\phi_t^f(U) \cap U \neq \varnothing$



- Since θ does not change along f_μ, it suffices to consider only the "R" part of f_μ. Hence, fix θ and let

$$h(R) = R[J^{-1}(R^{\mathrm{T}}\mu - \nu(\theta))]^{\times}$$

- Define a volume form Ω on $\mathrm{SO}(3)$ by

$$\Omega_R(Rv_1^{\times}, Rv_2^{\times}, Rv_3^{\times}) = v_1^{\mathrm{T}}(v_2 \times v_3)$$

- Flow of h conserves the volume form Ω

$$L_h\Omega \equiv 0$$

- Poincare's recurrence theorem $\Longrightarrow$ h is WPPS on $\mathrm{SO}(3)$

The drift vector field f_μ is WPPS on $\mathrm{SO}(3) \times \mathbb{T}^q$ for each μ

Accessibility $+$ WPPS $\Longrightarrow$ (global) controllability

- Dynamics are globally controllable on $\mathrm{SO}(3) \times \mathbb{T}^q$ for each μ
  - Given any two states having the same inertial angular momentum components, there exist gimbal angles that steer the spacecraft from one to the other
  - Controllability not affected by singular CMG configurations
- Controllability independent of the number and arrangement of CMGs

Practical Implications of Controllability

- What we have got:

$$\text{Steer } (R_\mathrm{i}, \omega_\mathrm{i}, \theta_\mathrm{i}) \text{ to } (R_\mathrm{f}, *, \theta_\mathrm{f})$$

- What we want:

$$\text{Steer } (R_\mathrm{i}, \omega_\mathrm{i}, \theta_\mathrm{i}) \text{ to } (R_\mathrm{f}, \omega_\mathrm{f}, *)$$

Given

- An initial $(R_\mathrm{i}, \omega_\mathrm{i}, \theta_\mathrm{i})$ and a desired final rotational state $(R_\mathrm{f}, \omega_\mathrm{f})$,

Does there exist

- A corresponding final CMG configuration θ_f such that
- $(R_\mathrm{i}, \omega_\mathrm{i}, \theta_\mathrm{i})$ and $(R_\mathrm{f}, \omega_\mathrm{f}, \theta_\mathrm{f})$ lie on the same angular momentum level set?

- The answer depends on the structure of angular momentum level set, and hence on the CMG array

Implications for Rate-to-Rest Maneuvers

- Suppose the total inertial angular momentum equals $\mu \in \mathbb{R}^3$
- If $(R, \theta) \in \mathrm{SO}(3) \times \mathbb{T}^q$ is a rest state on this angular momentum level set, then

$$R^{\mathrm{T}} \mu = \nu(\theta), \ \|\mu\| = \|\nu(\theta)\|$$

- The level set contains no rest state if

$$\|\mu\| > \max\{\|\nu(\theta)\| : \theta \in \mathbb{T}^q\} = \max\{\|v\| : v \in \mathcal{V}\}$$

  - CMG array gets *saturated* before rest state is achieved
  - No rest state possible inspite of controllability
- Spacecraft can be brought to rest in all desired attitudes if

$$\mathcal{V} \text{ contains a sphere of radius } \|\mu\|$$

- At least three CMGs needed

So Far

- Dynamics controllable on an angular momentum level set
  - The reachable set from any state is the angular momentum level set containing that state
- The result holds irrespective of singular CMG configurations or the construction of the CMG array
- Ability to steer the spacecraft to practically useful final states may still depend on the CMG array

- Let $\mathcal{S}$ = set of singular configurations
- Given $v \in \mathrm{S}^2$, let $\mathcal{S}_v$ = set of singular configurations with singular direction v

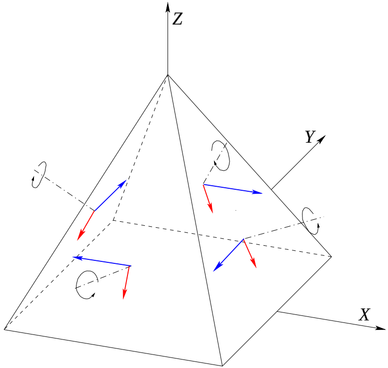
$$= \{\theta \in \mathbb{T}^q : v^{\mathrm{T}} \nu_i'(\theta_i) = 0,\ \forall\, i\}$$

- $\theta \in \mathcal{S}$ is an *external singularity* if $\theta \in \mathcal{S}_v$ for some $v \in \mathrm{S}^2$ and

$$\min_i v^{\mathrm{T}} \nu_i(\theta_i) > 0$$

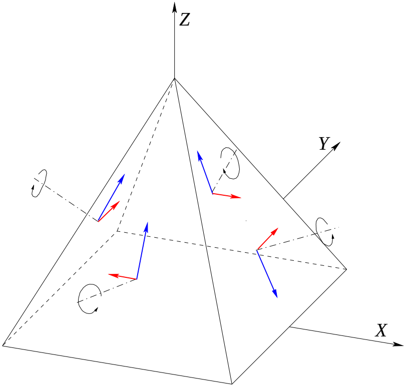
  - An external singularity is a strict global maximizer of $\theta \mapsto v^{\mathrm{T}} \nu(\theta)$ for some $v \in \mathrm{S}^2$
  - $\nu(\theta)$ lies on the momentum envelope

- Consider the function $\eta : \mathbb{T}^q \to \mathbb{R}$ given by $\eta(\theta) = \|\nu(\theta)\|^2$
- $\theta \in \mathbb{T}^q$ is a critically singular configuration if
  - $\theta \in \mathcal{S}$
  - θ is a critical point of η
- Denote $\mathcal{C} =$ set of critically singular configurations
  - If $\nu(\theta) = 0$ and $\theta \in \mathcal{S}$, then $\theta \in \mathcal{C}$
  - If $\nu(\theta) \neq 0$, then $\theta \in \mathcal{C}$ if and only if
    - $\nu(\theta)^{\mathrm{T}} \nu_i'(\theta_i) = 0 \; \forall \, i$
    - That is, the singular direction and $\nu(\theta)$ are linearly dependent
- A critically singular configuration may or may not be an external singularity
- Likewise, an external singularity may or may not be a critically singular configuration



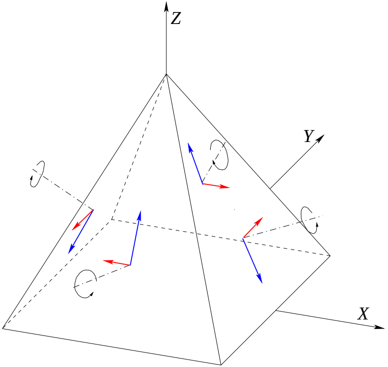
A non-singular configuration

- $\nu(\theta) = 0$
  - θ is a critical point of η
- $\theta \notin \mathcal{S}$
- Therefore $\theta \notin \mathcal{C}$



A non-critically singular configuration

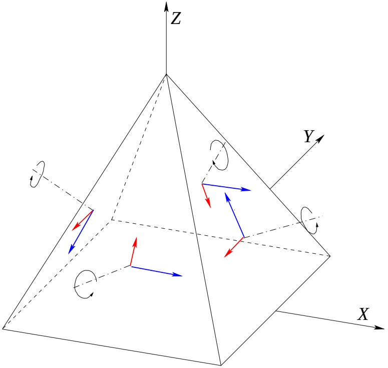
- All CMG torques in XY-plane
  - Singular direction along Z-axis
- Therefore $\theta \in \mathcal{S}$
- θ is an internal singularity
- $\nu(\theta)$ not along X-axis
  - Therefore $\theta \notin \mathcal{C}$



A critically singular configuration

$$\nu(\theta) = 0$$

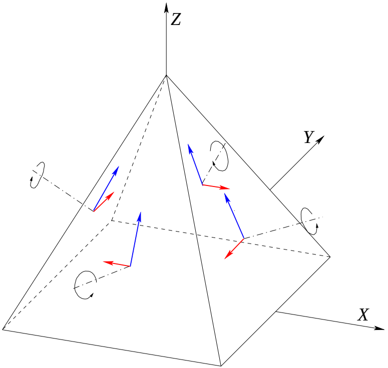
- $\nu(\theta) = 0$
  - θ is a critical point of η
- $\theta \in \mathcal{S}$
  - All CMG torques in XY-plane
- Therefore $\theta \in \mathcal{C}$
- θ is an internal singularity



A critically singular configuration

$$\nu(\theta) \neq 0$$

- $\nu(\theta)$ along X-axis
- All CMG torques in YZ-plane
  - Singular direction along X-axis
- Therefore $\theta \in \mathcal{C}$
- θ is an internal singularity



A critically singular external singularity

- $\nu(\theta)$ along Z-axis
- All CMG torques in XY-plane
  - Singular direction along Z-axis
- Therefore $\theta \in \mathcal{C}$
- θ is an external singularity
- θ is a local maximizer for η

Thank You