Spacecraft Attitude Control using CMGs: Local Controllability and Stabilizability

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Previous Talk

- Singular configurations and why people think they make control difficult
- Dynamics are globally as controllable as conservation of angular momentum permits
- Some care should be exercised when drawing conclusions in practical situations
- Two types of singular configurations: external and critically singular

This Talk

Do singular configurations affect

- Local controllability?
- Local stabilizability?

Recap of Notation and Terminology

- CMG angular momentum $\nu(\theta) = \nu_1(\theta_1) + \cdots + \nu_q(\theta_q)$
- CMG angular momentum magnitude defines a function $\eta(\theta) = \|\nu(\theta)\|^2$
- CMG configuration θ is *singular* with *singular direction* $v \in S^2$ if

$$v^T \nu_i'(\theta_i) = 0 \;\forall\; i$$

- A singular configuration θ with singular direction $v \in S^2$ is
  - An *external singularity* if

$$v^T \nu_i(\theta_i) > 0 \;\forall\; i$$

  - *Critically singular* if
    singular direction v and $\nu(\theta)$ are linearly dependent

Recap of the Dynamics

$$\begin{aligned} \dot{R} &= R[J^{-1}(R^{\mathrm{T}}\mu - \nu(\theta))]^{\times} \\ \dot{\theta} &= u \end{aligned}$$

$$\dot{y}(t) = \underbrace{f_\mu(y(t))}_{\text{drift}} + \underbrace{g_1(y(t))u_1(t) + \cdots + g_q(y(t))u_q(t)}_{\text{control}}$$

- Set of uncontrolled equilibria

$$\begin{aligned} \mathcal{E}_\mu &\overset{\text{def}}{=} \{x \in \mathrm{SO}(3) \times \mathbb{T}^q : f_\mu(x) = 0\} \\ &= \{(R, \theta) \in \mathrm{SO}(3) \times \mathbb{T}^q : R^{\mathrm{T}}\mu = \nu(\theta)\} \end{aligned}$$

Checking Controllability of the Linearization

$$\dot{y} = f(x) + g_1(x)u_1 + \cdots + g_q(x)u_q$$

$f(x_e) = 0,\ U$ an open neighborhood of $x,\ \phi : U \to \mathbb{R}^n$ a chart, $\phi(x_e) = 0$

Dynamics expressed in coordinates: $\hat{f} = \phi_* f,\ \hat{g}_i = \phi_* g_i$

Linearization: $\dot{\hat{x}} = A\hat{x} + b_1\hat{u}_1 + \cdots,\ A = \frac{\partial \hat{f}}{\partial \hat{x}}(0), b_i = \hat{g}_i(0)$

$$\begin{aligned} Ab_i &= \frac{\partial \hat{f}}{\partial \hat{x}}(0)\hat{g}_i(0) = \frac{\partial \hat{f}}{\partial \hat{x}}(0)\hat{g}_i(0) - \frac{\partial \hat{g}_i}{\partial \hat{x}}(0)\hat{f}(0) = -[\hat{f}, \hat{g}_i](0) \\ &= -[\phi_* f, \phi_* g_i](0) = -\phi_*[f, g](\phi(x_e)) = -\mathrm{T}_{x_e}\phi([f, g](x_e)) \end{aligned}$$

Similarly

$$A^2 b_i = \phi_*[f, [f, g]](\phi(x_e)) = \mathrm{T}_{x_e}\phi(\mathrm{ad}_f^2 g_i(x_e))$$

$$\text{rank } [B, A^2B, \ldots, A^{n-1}B] = \dim \text{span}\{\mathrm{ad}_f^k g_i(x_e) : k = 0, 1, \ldots, n-1\}$$

Lemma

Let $\mu \in \mathbb{R}^3$, $\theta \in \mathcal{C}$, and suppose $p = (R, \theta) \in \mathcal{E}_\mu$. Then

$$\mathrm{span}\{\mathrm{ad}^n_{f_\mu} g_i(p) : i, n \geq 1\} \subseteq \mathrm{span}\{(R(J^{-1}w)^\times, 0) : w \in \mathbb{R}^3, w^\mathrm{T}\nu(\theta) = 0\}$$

Result

Let $\mu \in \mathbb{R}^3$ and suppose $p = (R, \theta) \in \mathcal{E}_\mu$. Then the linearization of the dynamics at p is controllable if and only if θ is not a critically singular configuration.

Corollary

If $\|\mu\|^2$ is a regular value of the function $\eta(\cdot) = \|\nu(\cdot)\|^2$, then the dynamics have a controllable linearization at every equilibrium in $\mathcal{E}_\mu$

- Gimbal rates are measurable functions of time taking values in the *polydisk*

$$\mathcal{H}_\rho \stackrel{\text{def}}{=} \{u \in \mathbb{R}^q : |u_i| \le \rho_i,\ \forall\ i\},\ \ \rho_i > 0$$

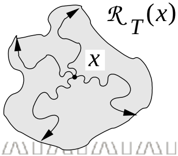
- Reachable set

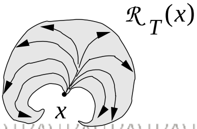
$\mathcal{R}_T(x)$ = set of states reached in time $\le T$ by starting from $x \in \mathrm{SO}(3) \times \mathbb{T}^q$ at time 0 and using gimbal rates lying in $\mathcal{H}_\rho$

- Dynamics are *small-time locally controllable (STLC)* if

$$x \in \operatorname{int} \mathcal{R}_T(x)\ \forall\ T > 0$$

STLC





Not STLC

Linearization controllable $\Longrightarrow$ STLC

Result

Dynamics are STLC at an equilibrium if the CMG array at that equilibrium is not in a critically singular configuration

Corollary

- Dynamics are STLC at all equilibria on an angular momentum level set corresponding to a regular value of the function η
- Dynamics are STLC at all equilibria on almost all angular momentum level sets (Sard's theorem)

Sufficient Conditions for STLC: A First Attempt

- Assign nonnegative weights $l_0, l_1, \ldots, l_q$ to $f_\mu, g_1, \ldots, g_q$, resp.
- A bracket B involving $f_\mu, g_1, \ldots, g_q$
  - Is *bad* if it contains f_μ an odd number of times and each g_i an even number of times
  - Is *good* otherwise
  - Has l-degree $l_0|B|_0 + l_1|B|_1 + \cdots + l_q|B|_q$
- STLC holds at an equilibrium x_e if
  - Every bad bracket evaluated at x_e is in the span of good brackets of lower l-degree
- Stronger than the Bianchini-Stefani condition
- Fails to hold at equilibria involving certain critically singular configurations

- $\mathrm{Lie}(\xi)$ = free Lie algebra in indeterminates $\xi = \{\xi_0, \xi_1, \ldots, \xi_q\}$
- Containing real linear combinations of formal Lie brackets involving $\{\xi_0, \xi_1, \ldots, \xi_q\}$ like
  - ξ_0, ξ_1, $[\xi_0, [\xi_1, \xi_2]]$, $3\xi_3 + 2[\xi_0, \xi_1] + 1.43[\xi_0, [\xi_1, \xi_2]]$
- $\mathrm{Lie}_0(\xi)$ = subalgebra generated by elements of the form $\mathrm{ad}^k_{\xi_0} B$, $B \in \mathrm{Lie}(\xi)$
- $\mathrm{Lie}_0(\xi)$ contains real linear combinations of elements like
  - $[\xi_0, \xi_1]$, $[\xi_0, B]$, $[\mathrm{ad}^k_{\xi_0} B, C]$, $B, C \in \mathrm{Lie}(\xi)$
- $\mathrm{Lie}_0(\xi)$ is the smallest Lie subalgebra of $\mathrm{Lie}(\xi)$ containing $\{\xi_1, \ldots, \xi_q\}$ and closed under brackets with ξ_0

TATA

- Admissible weight vector: $l = [l_0, l_1, \dots, l_q]^{\mathrm{T}} \in \mathbb{R}^n$ such that $l_i \geq l_0 \geq 0$
  - Running example: $q = 3$, $l_0 = 1$, $l_2 = 1.5$, $l_1 = l_3 = 2$
- $|B|_i =$ no. of times ξ_i appears in the bracket $B \in \mathrm{Lie}(\xi)$
- l-degree of bracket B equals $l_0|B|_0 + \cdots + l_q|B|_q$
  - $[\xi_0, [\xi_1, \xi_2]]$ has l-degree 4.5, $[[\xi_0, \xi_1], [\xi_1, \xi_2]]$ has l-degree 6.5
- $B \in \mathrm{Lie}(\xi)$ is *l-homogeneous* if it is a combination of brackets having the same l-degree
  - $2.3[\xi_0, [\xi_1, \xi_2]] + 6.31\mathrm{ad}^3_{\xi_0}\xi_2$ is l-homogeneous of degree 4.5
  - $2.3[\xi_0, [\xi_1, \xi_2]] + 6.31\mathrm{ad}^3_{\xi_0}\xi_1$ is not l-homogeneous
- $\mathcal{V}_k =$ subspace of $\mathrm{Lie}_0(\xi)$ generated by brackets having l-degree $\leq k$
  - $2.3[\xi_0, [\xi_1, \xi_2]] + 6.31\mathrm{ad}^3_{\xi_0}\xi_2 \in \mathcal{V}_{4.5} \subseteq \mathcal{V}_5$
  - $2.3[\xi_0, [\xi_1, \xi_2]] + 6.31\mathrm{ad}^3_{\xi_0}\xi_1 \notin \mathcal{V}_{4.5}$, but $\in \mathcal{V}_5$

TATA

- The bracket $B \in \mathrm{Lie}_0(\xi)$ is *bad* if $|B|_0$ is odd and $|B|_i$ is even for each $i > 0$
  - $[\xi_2, [\xi_0, \xi_2]]$, $[\xi_1, \mathrm{ad}^3_{\xi_0}\xi_1]$ are bad, $[\xi_1, [\xi_0, \xi_2]]$, $[\xi_1, \mathrm{ad}^2_{\xi_0}\xi_1]$ are not
- $\mathcal{B}$ = subspace of $\mathrm{Lie}_0(\xi)$ generated by bad brackets
- $\mathcal{B}_S$ = subset of elements of $\mathcal{B}$ that remain unchanged whenever ξ_i and ξ_j are interchanged for any pair $i, j > 0$ such that $l_i = l_j$
  - $[\xi_0, \xi_2] + a[\xi_2, [\xi_0, \xi_1]] + b[\xi_2, [\xi_0, \xi_3]] \in \mathcal{B}_S$ if $a = b$, $\notin \mathcal{B}_S$ otherwise
  - We can “symmetrize” any bad bracket to get an element of $\mathcal{B}_S$
- Set $\mathcal{B}^*_S$ of *l-obstructions* is the smallest Lie algebra containing $\mathcal{B}_S$ and closed under Lie brackets with ξ_0
  - $\mathcal{B}^*_S$ = Lie subalgebra generated by elements of the form $\mathrm{ad}^k_{\xi_0} B$, $B \in \mathcal{B}_S$

- Given a bracket $B \in \mathrm{Lie}(\xi)$, $p \in \mathrm{SO}(3) \times \mathbb{T}^q$ and a set of vector fields $\mathbf{h} = \{h_0, h_1, \ldots, h_q\}$ on $\mathrm{SO}(3) \times \mathbb{T}^q$,

$$\begin{aligned} \mathrm{Ev}^{\mathbf{h}}(B) &= \text{vector field obtained by replacing } \xi_i \text{ with } h_i \\ \mathrm{Ev}^{\mathbf{h}}_p(B) &= \text{tangent vector at } p \text{ obtained by evaluating } \mathrm{Ev}^{\mathbf{h}}(B) \text{ at } p \\ \mathcal{V}^{\mathbf{h}}_k(p) &= \{\mathrm{Ev}^{\mathbf{h}}_p(B) : B \in \mathcal{V}_k\} \end{aligned}$$

- An l-homogeneous element $B \in \mathcal{B}^*_{\mathrm{S}}$ is **h**-l-neutralized at p if there exists $k < l$-degree of B such that

$$\mathrm{Ev}^{\mathbf{h}}_p(B) \in \mathcal{V}^{\mathbf{h}}_k(p)$$

The Bianchini-Stefani Condition

The dynamics are STLC at $p \in \mathcal{E}_\mu$ under the input constraint $u \in \mathcal{H}_\rho$ if there exist

1. a nonnegative k
2. an admissible weight vector l

such that

1. every l-homogeneous element of $\mathcal{B}^*_{\mathrm{S}}$ of l-degree $\leq k$ is $\mathbf{h}$-l-neutralized at p and
2. $\mathcal{V}^{\mathbf{h}}_k(p)$ equals the tangent space at p

for $\mathbf{h} = \{f_\mu, g_1, \ldots, g_q\}$

- Condition does not involve the constraint parameters ρ

R. M. Bianchini and G. Stefani, "Controllability along a trajectory: a variational approach," *SIAM J. Contr. Optim.*, 1993

F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems*, Springer, 2005

Applying the Bianchini-Stefani Condition

- Consider $p = (R, \theta) \in \mathcal{E}_\mu$ such that $\theta \in \mathcal{C}$ and $\|\nu(\theta)\|(= \|\mu\|) = 0$
- Choose all weights $= 1$
- Two lowest possible degrees for a bad bracket are 3 and 5
  - A bad bracket of degree 3 is necessarily of the form $B = [\xi_i, [\xi_0, \xi_i]]$
  - Corresponding symmetrized element is $B_S = \sum_{i=1}^{q} [\xi_i, [\xi_0, \xi_i]]$

$$\mathrm{Ev}^{\mathbf{h}}(B) = [g_i, [f_\mu, g_i]] = (-R(J^{-1}\nu_i)^\times, 0)$$

$$\therefore \mathrm{Ev}_p^{\mathbf{h}}(B_S) = (-R(J^{-1}\nu(\theta))^\times, 0) = 0$$

  - Similarly, bad brackets of degree 5 also vanish after symmetrization
- $\mathcal{V}_5^{\mathbf{h}}(p)$ contains the $3 + q$ linearly independent tangent vectors $g_i(p)$ and

$$[f_\mu, g_1](p), \ \underbrace{[g_1, [f_\mu, g_1]]}_{\text{bad bracket}}(p), \ [[g_1, [f_\mu, g_1]], [f_\mu, g_1]](p)$$

- STLC follows

Main result

Let $\mu \in \mathbb{R}^3$, and suppose $p = (R, \theta) \in \mathcal{E}_\mu$ is such that $\theta \in \mathcal{C}$. If any one of the following three conditions hold, then the dynamics are STLC at p subject to $u \in \mathcal{H}_\rho$

$$\nu(\theta) = 0$$

$$\min_i \nu(\theta)^{\mathrm{T}} \nu_i(\theta_i) < 0$$

$$\min_i \nu(\theta)^{\mathrm{T}} \nu_i(\theta_i) = 0,\ \dim \operatorname{span}\{\nu_i'(\theta_i) : i \text{ s.t. } \nu(\theta)^{\mathrm{T}} \nu_i(\theta_i) = 0\} = 2$$

- Second condition $\iff$ θ is not an external singularity
- Cases not covered

$$\min_i \nu(\theta)^{\mathrm{T}} \nu_i(\theta_i) > 0 \text{ (external singularity)}$$

$$\min_i \nu(\theta)^{\mathrm{T}} \nu_i(\theta_i) = 0,\ \dim \operatorname{span}\{\nu_i'(\theta_i) : i \text{ s.t. } \nu(\theta)^{\mathrm{T}} \nu_i(\theta_i) = 0\} = 1$$

Stefani's Necessary Condition for STLC

Suppose $y : [0, \hat{T}] \to \mathrm{SO}(3) \times \mathbb{T}^q$ is a solution of the uncontrolled system and $\gamma : [0, \hat{T}] \to \mathrm{T}^*(\mathrm{SO}(3) \times \mathbb{T}^q)$ is a solution of the adjoint system of the uncontrolled system such that $\gamma(t) \in \mathrm{T}^*_{y(t)}\mathcal{N}$, $t \in [0, \hat{T}]$ and

1. $\gamma(t)(\mathrm{ad}^k_{f_\mu} g_i(y(t))) = 0$, $t \in [0, \hat{T}]$, for every i, k
2. $L \in \mathbb{R}^{q \times q}$ defined by $L_{ij} = \gamma(0)\left([[f_\mu, g_i], g_j](y(0))\right)$ is positive definite
3. $\ldots$

Then there exists $T \in (0, \hat{T}]$ such that, for all $t \in [0, T]$,

$$y(t) \text{ lies on the boundary of } \mathcal{R}_t(y(t))$$

- Consequence of a sufficient condition for extremality
- Idea: apply with y and γ constant solutions

G. Stefani, "A sufficient condition for extremality," *Analysis and Optimization of Systems*, LNCIS # 111, Springer, 1988

Coordinate-free description

The adjoint system of the vector field f_μ is the Hamiltonian vector field on $\mathrm{T}^*(\mathrm{SO}(3)\times\mathbb{T}^q)$ having the Hamiltonian function defined by

$$H(\Lambda)=\Lambda(f_\mu(x)),\ \Lambda\in\mathrm{T}^*(\mathrm{SO}(3)\times\mathbb{T}^q),\ x=\pi^*(\Lambda)$$

Coordinate description

$$\begin{array}{lrcll} \text{System:} & \dot{x}(t) & = & f(x(t)), & f:\mathbb{R}^n\to\mathbb{R}^n, \\ \text{Adjoint:} & \dot{\gamma}^{\mathrm{T}}(t) & = & -\gamma^{\mathrm{T}}(t)\frac{\partial f}{\partial x}(x(t)) & \end{array}$$

If $x\equiv x_{\mathrm{e}}$ is a constant solution, then the adjoint solution $\gamma(\cdot)$

- Is constant iff it is a left-null vector of the system linearization at x_{e}
- Satisfies $\gamma(t)(\mathrm{ad}^k_{f_\mu}g_i(x_{\mathrm{e}}))=0$ iff it lies in the left null space of the controllability matrix of the system linearization at x_{e}
- Satisfies both of the above only if the linearization at x_{e} has an uncontrollable eigenvalue at 0

Main result

Suppose $\mu \neq 0$. Let $\theta \in \mathcal{C}$ be such that

$$\min\{\nu(\theta)^{\mathrm{T}}\nu_i(\theta_i) : i \in \mathbb{I}_q\} > 0, \text{ (external singularity)}$$

and let $p = (R_{\mathrm{e}}, \theta_{\mathrm{e}}) \in \mathcal{E}_\mu$. Then the dynamics are not STLC at p.

- $\gamma \equiv (R_{\mathrm{e}}(J\nu(\theta_{\mathrm{e}}))^\times, 0) \in \mathrm{T}_{R_{\mathrm{e}}}(\mathrm{SO}(3) \times \mathbb{T}^q)$ is an adjoint solution
- Recall that

$$\mathrm{ad}^n_{f_\mu} g_i(p) \in \mathrm{span}\{(R_{\mathrm{e}}(J^{-1}w)^\times, 0) : w \in \mathbb{R}^3, w^{\mathrm{T}}\nu(\theta_{\mathrm{e}}) = 0\}$$

- Matrix L is diagonal with $L_{ii} = \nu(\theta_{\mathrm{e}})^{\mathrm{T}}\nu_i(\theta_{\mathrm{e}}) > 0$
- Result follows from Stefani's condition

- Dynamics are not STLC at $p = (R, \theta) \in \mathcal{E}_\mu$ if θ is a local maximizer for η
  - Second-order necessary conditions for a local maximum $\implies$ Hessian is nonnegative definite $\implies \nu(\theta)^{\mathrm{T}} \nu_i(\theta_i) > 0$ for all i
- In case of only one CMG, dynamics are STLC at no equilibrium
  - η is a constant function, and every configuration is a local maximizer
- Can we identify small-time unreachable states?

Symmetry

- Isotropy group of μ (assumed $\neq 0$)

$$\mathcal{I}_\mu \stackrel{\text{def}}{=} \{S \in \mathrm{SO}(3) : S\mu = \mu\} = \{\mathrm{e}^{\alpha\mu^\times} : \alpha \in \mathbb{R}\}$$

- $\mathcal{I}_\mu$ acts on $\mathrm{SO}(3) \times \mathbb{T}^q$ through the action

$$\Phi_S^\mu(x) = (SR, \theta), \; x = (R, \theta)$$

- If $(R(\cdot), \theta(\cdot))$ is a solution, then so is $(SR(\cdot), \theta(\cdot))$ for each $S \in \mathcal{I}_\mu$
  - Dynamics on $\mathrm{SO}(3) \times \mathbb{T}^q$ are invariant under the action of $\mathcal{I}_\mu$

$$\mathcal{R}_T(\Phi_S^\mu(x)) = \Phi_S^\mu(\mathcal{R}_T(x))$$

- Define "projection" $\phi_\mu : \mathrm{SO}(3) \times \mathbb{T}^q \to \mathrm{S}^2 \times \mathbb{T}^q$

$$\phi_\mu(x) = (\|\mu\|^{-1} R^{\mathrm{T}} \mu, \theta), \; x = (R, \theta)$$

- Fiber over $x^{\mathrm{r}} \in \mathrm{S}^2 \times \theta$ is an orbit of $\mathcal{I}_\mu$

- Reduced state $\mathrm{S}^2 \times \mathbb{T}^q \ni (\xi, \theta) \stackrel{\text{def}}{=} x^{\mathrm{r}} = \phi_\mu(x) = (\|\mu\|^{-1} R^{\mathrm{T}} \mu, \theta)$
- Reduced dynamics

$$\dot{\xi} = \xi \times [J^{-1}\{\|\mu\|\xi - \nu(\theta)\}], \; \dot{\theta} = u$$

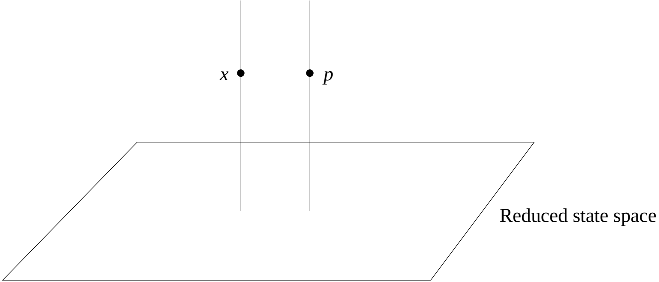
- Easy consequences

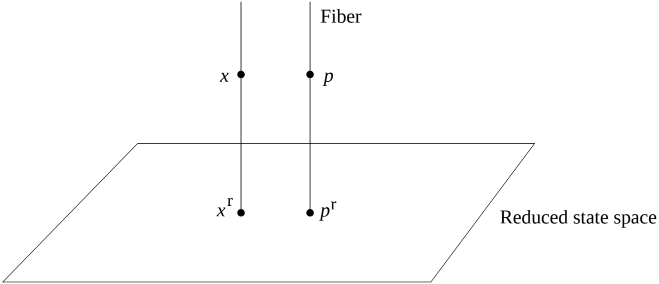
$$\mathcal{R}_T^{\mathrm{r}}(\phi_\mu(x)) = \phi_\mu(\mathcal{R}_T(x)), \; \phi_\mu(\mathcal{E}_\mu) \subseteq \mathcal{E}_\mu^{\mathrm{r}}$$

STLC of reduced dynamics

Suppose $\mu \neq 0$. If $p \in \mathcal{E}_\mu$, then the linearization of the reduced dynamics at $p^{\mathrm{r}} \stackrel{\text{def}}{=} \phi_\mu(p)$ are controllable. Consequently, the reduced dynamics are STLC at p^{r}.

Implications for the Full Dynamics




Fiber
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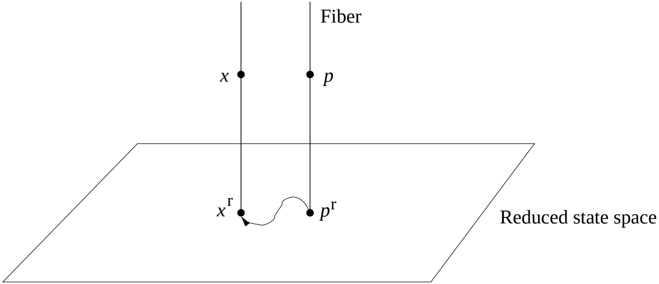
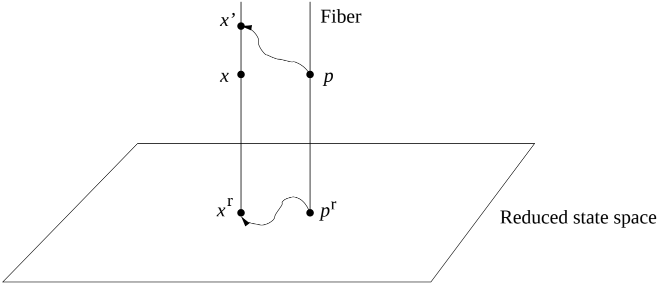
p^r
Reduced state space

Implications for the Full Dynamics



Implications for the Full Dynamics



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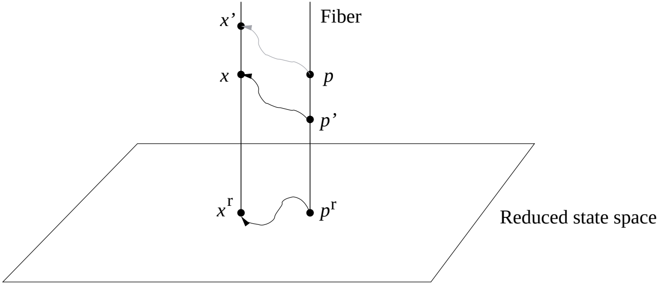
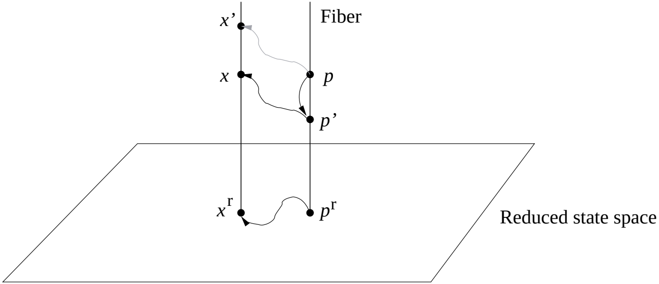


Implications for the Full Dynamics



- If all nearby points on the fiber can be reached in small time, then the full dynamics must be STLC

Small-Time Unreachable States

Suppose $\mu \neq 0$. Let θ be a critically singular external singularity, and assume $p = (R, \theta)$ is an equilibrium point. Then there exist $T > 0$ and a sequence of angles $\{\alpha_n\}_{n=1}^{\infty}$ converging to 0 in $(-\pi, \pi)$ such that

$$(\exp(\alpha_n \mu^{\times})R, \theta) \notin \mathcal{R}_T(p)$$

- There exist arbitrarily small rotations about the inertial angular momentum vector (equivalently, the singular direction) that cannot be achieved in time less than T with zero net change in the gimbal angles

Results

Suppose $p = (R_e, \theta_e) \in \mathcal{E}_\mu$.

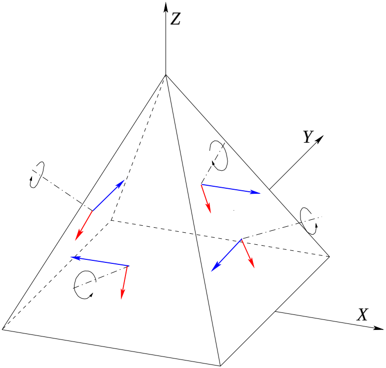
1. If θ_e is not a critically singular configuration, then p is locally asymptotically stabilizable (linearization is controllable)
2. If θ_e either yields a local maximum or a nonzero local minimum for η, then p is not locally asymptotically stabilizable
   - Single CMG $\implies$ no equilibrium is stabilizable

- Choose neighborhood U of p such that

$$(R^T\mu)^T\nu(\theta_e) > 0 < \nu(\theta_e)^T\nu(\theta) \;\forall\; (R, \theta) \in U$$

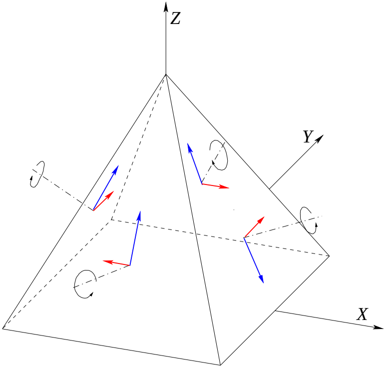
- There exists $(R, \theta) \in U$ and $\epsilon < 0$ such that $R^T\mu - \nu(\theta) = \epsilon\nu(\theta_e) \implies$

$$\begin{aligned}\|\nu(\theta_e)\|^2 - \|\nu(\theta)\|^2 &= \|R^T\mu\|^2 - \|\nu(\theta)\|^2 \\ &= (R^T\mu + \nu(\theta))^T(R^T\mu - \nu(\theta)) = \epsilon(R^T\mu + \nu(\theta))\nu(\theta_e) < 0\end{aligned}$$



A non-singular configuration

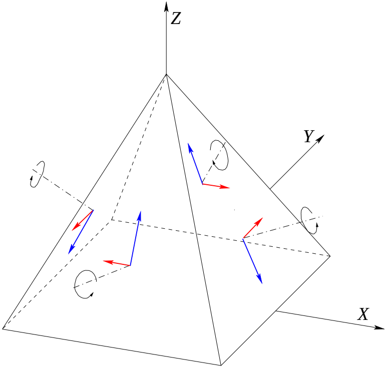
- Linearization controllable
- STLC and stabilizability hold



A non-critically singular configuration

$\nu(\theta) \neq 0$

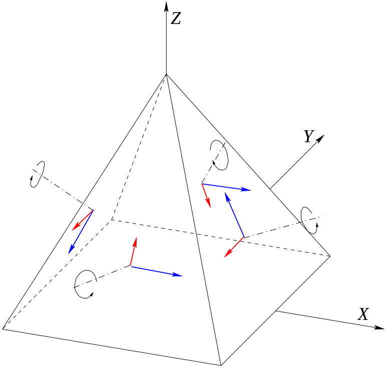
- Linearization controllable
- STLC and stabilizability hold



A critically singular configuration

$$\nu(\theta) = 0$$

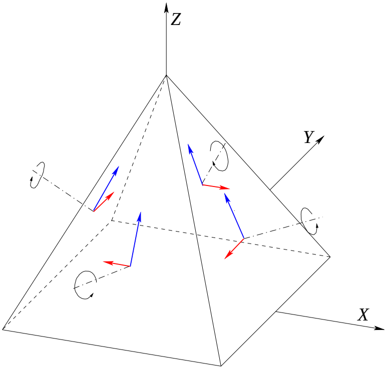
- Linearization uncontrollable
- STLC holds



A critically singular configuration

$$\nu(\theta) \neq 0$$

- Internal singularity
- Linearization uncontrollable
- STLC holds



A critically singular external singularity

- θ is a local maximizer for η
- STLC and stabilizability fail

Summary

- STLC and stabilizability depend on the nature of the singular configuration
  - Non-critically singular configurations pose no problems for STLC, stabilizability
  - Critically singular configurations that are not external singularities pose no problems for STLC
  - Critically singular external singularities $\Longrightarrow$ no STLC
    - Small rotations about the singular direction not achievable in small time
    - Includes local maximizers of CMG angular momentum magnitude as special cases
    - Includes single CMG as a special case
  - Local maximizer of CMG angular momentum magnitude $\Longrightarrow$ no stabilizability

Thank You