CONTROL IN GEOMETRIC MECHANICS

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PLAN OF THE PRESENTATION

- ***Poisson reduction***
- ***Lie-Poisson reduction***
- ***Hamilton's variational principle***
- ***Euler-Poincaré reduction***
- ***The free top (Euler top)***
- ***Affine Euler-Poincaré reduction***

- *The heavy top*
- *Continuum mechanics setting*
- *Fixed boundary barotropic fluids*
- *Elasticity*
- *Symmetric representation of the free rigid body equations*
- *Clebsch optimal control*

POISSON REDUCTION

$\phi : G \times P \to P$ Lie group acting canonically on a Poisson manifold $(P, \{\, , \}_P)$:

$$\{F \circ \Phi_g, H \circ \Phi_g\}_P = \{F, H\}_P \circ \Phi_g, \quad \forall g \in G, \quad F, H \in C^\infty(P).$$

Assume that the orbit space P/G is a smooth manifold and the quotient projection $\pi : P \to P/G$ a surjective submersion (e.g., G-action is proper and free, or proper with all isotropy groups conjugate). Then there exists a unique Poisson bracket $\{\cdot, \cdot\}_{P/G}$ on P/G relative to which π is a Poisson map. The Poisson bracket on P/G is given in the following way. If $\widehat{F}, \widehat{H} \in \mathcal{F}(P/G)$, then $\widehat{F} \circ \pi, \widehat{H} \circ \pi \in \mathcal{F}(P)$ are G-invariant functions and, due to the fact that the action is canonical, their Poisson bracket $\{\widehat{F} \circ \pi, \widehat{H} \circ \pi\}_P$ is also G-invariant. Therefore, this function descends to a smooth function on the quotient P/G; this is, by definition, $\{\widehat{F}, \widehat{H}\}_{P/G}$ and we have, by construction,

$$\{\widehat{F} \circ \pi, \widehat{H} \circ \pi\}_P = \{\widehat{F}, \widehat{H}\}_{P/G} \circ \pi.$$

LIE-POISSON REDUCTION

$P = T^*G$ and the G-action the lift of left translation. *Then* $\mathbf{J}_R : \alpha_g \mapsto T_e^* L_g(\alpha_g) \in \mathfrak{g}^*$ *drops to a diffeomorphism* $(T^*G)/G \xrightarrow{\sim} \mathfrak{g}^*$ *and the quotient bracket pushes forward to the Lie-Poisson bracket*

$$\{f, h\}(\mu) = -\left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu}\right]\right\rangle.$$

If $H : T^*G \to \mathbb{R}$ *is a left* G*-invariant function its restriction* $h := H|_{\mathfrak{g}^*}$ *satisfies* $H = h \circ \mathbf{J}_R$. *The flow* F_t *of* X_H *on* T^*G *and the flow,* F_t^L *of* X_h *on* $\mathfrak{g}^*_-$ *are related by* $\mathbf{J}_R \circ F_t = F_t^L \circ \mathbf{J}_R$.

For the right action use $\mathbf{J}_L : \alpha_g \in T^*G \mapsto T_e^* R_g(\alpha_g) \in \mathfrak{g}^*$ and $+$ in front of the Lie-Poisson bracket.

This is the basic example of a dual pair.

Find F_t**:** First solve $\dot{\mu} = \mathrm{ad}^*_{\frac{\delta h}{\delta \mu}} \mu$, $\mu(0)$ given, then $\dot{g} = g \frac{\delta h}{\delta \mu}$, $g(0) = e$

HAMILTON'S PRINCIPLE

Q a manifold, called ***configuration space***

$\tau_Q : TQ \to Q$ its tangent bundle, called ***state space***

Recall the "physics definition" of the tangent bundle:
$t \in [0,1] \overset{C^\infty}{\longmapsto} q_i(t) \in Q$, $i = 1,2$, equivalent $\overset{\text{def}}{\Longleftrightarrow} q_1(0) = q_2(0) = q$, $\dot{q}_1(0) = \dot{q}_2(0)$ in a local chart (hence all local charts) at q.
An equivalence class is, by definition, a tangent vector $v_q \equiv (q, \dot{q})$ to Q at q. All such tangent vectors form the tangent space T_qQ. Then one proves: T_qQ is a vector space of dimension equal to $\dim Q$ and $\tau_Q : TQ := \cup_{q\in Q} T_qQ \ni v_q \mapsto q \in Q$ is a vector bundle.

$L : TQ \to \mathbb{R}$ a smooth function, called the ***Lagrangian***. It is given by physical considerations. For ***classical mechanical systems***, Q is a Riemannian manifold, $L(v_q) = \frac{1}{2}\|v_q\|^2 - V(q)$, $\frac{1}{2}\|v_q\|^2$ is the ***kinetic energy***, $V : Q \to \mathbb{R}$ is the ***potential energy***.

The ***action*** $\mathfrak{S}(L) : \Omega(Q; q_0, q_1) \to \mathbb{R}$

$$\mathfrak{S}(L)(q(\cdot)) := \int_{t_0}^{t_1} L(q(t), \dot{q}(t))\, dt$$

of the Lagrangian $L : TQ \to \mathbb{R}$ is defined on the ***space of paths with fixed endpoints***

$$\Omega(Q; q_0, q_1) := \{q(\cdot) \in C^1([t_0, t_1], Q) \mid q(t_0) = q_0,\ q(t_1) = q_1\}.$$

Here one has to specify regularity of the path - formally it is C^1.

$q(t, \lambda)$, $(t, \lambda) \in [t_0, t_1] \times [-\varepsilon, \varepsilon]$ is a ***deformation*** of $q(t) \in \Omega(Q; q_0, q_1)$ if $q(\cdot, \lambda) \in \Omega(Q; q_0, q_1)$, for all $\lambda \in [-\varepsilon, \varepsilon]$ (so $q(t_i, \lambda) = q_i$, $i = 0, 1$) and $q(t, 0) = q(t)$.

$$\delta q(t) := \left.\frac{\partial}{\partial \lambda}\right|_{\lambda=0} q(t, \lambda) \in T_{q(t)}Q, \qquad \delta q : [t_0, t_1] \to TQ$$

is the ***variation*** of the deformation $q(t, \lambda)$; note $\delta q(t_i) = 0$ for $i = 0, 1$.

Under good regularity conditions on the curves $q : [t_0, t_1] \to Q$, $\Omega(Q; q_0, q_1)$ is a smooth manifold, so one can ask what are the critical points of the action $\mathfrak{S}(L)$.

Hamilton's Principle: $\mathbf{d}\mathfrak{S}(L)(q(\cdot)) = 0 \Longleftrightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}) = \frac{\partial L}{\partial q^i}(q, \dot{q})$

$$\mathbf{d}\mathfrak{S}(L)(q(\cdot)) \cdot \delta q(\cdot) := \frac{d}{d\lambda}\mathfrak{S}(L)\left(q(\cdot, \lambda)\right)\Big|_{\lambda=0}.$$

Remark: This has intrinsic sense if one introduces the second order tangent bundle $T^{(2)}Q \to Q := J^2_0(\mathbb{R}, Q)$, the 2-jets of curves from $\mathbb{R} \to Q$ based at $0 \in \mathbb{R}$ (Bourbaki). Explanation of this concept:

$t \in [0,1] \overset{C^\infty}{\longmapsto} q_i(t) \in Q$, $i = 1, 2$, are equivalent $\overset{\text{def}}{\Longleftrightarrow} q_1(0) = q_2(0) = q$, $\dot{q}_1(0) = \dot{q}_2(0) \in T_qQ$, and $\ddot{q}_1(0) = \ddot{q}_2(0)$ in a local chart (hence all local charts) at q.

Equivalence classes, denoted by $(q, \dot{q}, \ddot{q})$, are called ***second order jets*** at $q \in Q$; all such jets is denoted by $T^{(2)}_q Q$. Define

$$T^{(2)}Q := \cup_{q \in Q} T^{(2)}_q Q$$

$\tau_Q^{(2)} : T^{(2)}Q \ni (q, \dot{q}, \ddot{q}) \mapsto q \in Q$ is a fiber (not a vector) bundle.

Global version of Hamilton's Principle: $\exists!\ \mathcal{EL}(L) : T^{(2)}Q \to T^*Q$ bundle map over Q, the ***Euler–Lagrange operator***, such that, for any deformation $q(t, \lambda)$, keeping the endpoints fixed, we have

$$\mathbf{d}\mathfrak{S}(L)(q(\cdot)) \cdot \delta q(\cdot) = \int_{t_0}^{t_1} \mathcal{EL}(L)(q(t), \dot{q}(t), \ddot{q}(t)) \cdot \delta q(t) dt,$$

In standard local coordinates, $\mathcal{EL}(L) : T^{(2)}Q \to T^*Q$ has the form

$$\mathcal{EL}(L)_i(q, \dot{q}, \ddot{q})\, dq^i = \left(\frac{\partial L}{\partial q^i}(q, \dot{q}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}) \right) dq^i.$$

Convention: In second term on the right hand side formally apply the chain rule then replace dq/dt by $\dot{q}$ and $d\dot{q}/dt$ by $\ddot{q}$.

So $\mathbf{d}\mathfrak{S}(L)(q(\cdot)) = 0 \iff$ the ***Euler-Lagrange equations*** hold:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}) = \frac{\partial L}{\partial q^i}(q, \dot{q})$$

Symmetries induce conserved quantities, the momentum maps. This enables one to eliminate variables. The best situation is when one has $\dim Q$ functions that Poisson commute on T^*Q and are functionally independent, i.e., their differentials are linearly independent almost everywhere. Then the system is ***completely integrable*** and can be solved, in many cases explicitly (algebraic geometry methods if applicable, or finding explicit action-angle variables).

QUESTION: What happens if symmetries are present? Clearly there is a problem since $(TQ)/G$ is *not* a tangent bundle.

There is a general answer: $(TQ)/G \cong_{\mathcal{A}} T(Q/G) \oplus \mathrm{Ad}\, Q$ (Cendra, Marsden, Ratiu, Memoirs of the AMS 2001).

There is no time to do the full theory, so we shall do only the case $Q =$ a Lie group and $G =$ certain subgroups various forms associated to it. Then we shall discuss many concrete applications.

EULER-POINCARÉ REDUCTION

Poincaré 1901: Left (right) invariant Lagrangian $L : TG \to \mathbb{R}$, $l := L|_{\mathfrak{g}} : \mathfrak{g} \to \mathbb{R}$. For $g(t) \in G$, let $\xi(t) = g(t)^{-1}\dot{g}(t)$ $\left(\dot{g}(t)g(t)^{-1} \in \mathfrak{g}\right)$. Equivalences:

(i) $\mathcal{EL}(L)(g(t), \dot{g}(t), \ddot{g}(t)) = 0$.

(ii) The variational principle

$$\delta \int_{t_0}^{t_1} L(g(t), \dot{g}(t))dt = 0$$

holds, for variations with fixed endpoints.

(iii) The ***Euler-Poincaré equations*** hold: $\frac{d}{dt}\frac{\delta l}{\delta \xi} = \pm \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}$.

(iv) The ***Euler-Poincaré variational principle***

$$\delta \int_{t_0}^{t_1} l(\xi(t))dt = 0$$

holds on $\mathfrak{g}$, for variations $\delta\xi = \dot{\eta} \pm [\xi, \eta]$, where $\eta(t)$ is an arbitrary path in $\mathfrak{g}$ that vanishes at the endpoints, i.e $\eta(t_0) = \eta(t_1) = 0$.

There is a unique map $\mathcal{EP} : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}^*$ such that for any deformation $\xi(t, \lambda) = g(t, \lambda)^{-1}\dot{g}(t, \lambda) \in \mathfrak{g}$ induced by a deformation $g(t, \lambda) \in G$ of the curve $g(t)$ keeping the endpoints fixed, thus $\delta g(t_i) = 0$ for $i = 1, 2$, we have

$$\delta \int_{t_0}^{t_1} l(\xi(t))dt = \int_{t_0}^{t_1} \left\langle \mathcal{EP}(l)(\xi(t), \dot{\xi}(t)), \eta(t) \right\rangle dt$$

for $\delta\xi(t) := \frac{\partial \xi(t,\lambda}{\partial \lambda}\Big|_{\lambda=0} = \dot{\eta}(t) \pm [\xi(t), \eta(t)]$ $\mathcal{EP}(l)$ is the

Euler-Poincaré operator:

$$\mathcal{EP}(l)(\xi, \dot{\xi}) = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} \mp \frac{d}{dt}\frac{\delta l}{\delta \xi}$$

where on the right hand side the time derivative is taken formally using the chain rule and $d\xi/dt$ is replaced at the end of the computation everywhere by $\dot{\xi}$.

Legendre transformation: $\mathfrak{g} \ni \xi \mapsto \mu := \delta l/\delta \xi \in \mathfrak{g}^*$. If invertibile define the Hamiltonian $h(\mu) = \langle \mu, \xi \rangle - l(\xi)$ on $\mathfrak{g}^*$ and then the Euler-Poincaré equations become the Lie-Poisson equations $\frac{d\mu}{dt} = \pm \, \mathrm{ad}^*_{\delta h/\delta \mu} \mu$.

Geometry has been replaced by analysis! Extend the method of the Calculus of Variations to such variational principles. I am not aware of any serious analysis results for such variational principles.

Reconstruction

Solve the Euler-Lagrange equations for a left (right) invariant $L : TG \to \mathbb{R}$ knowing the solution of the EP equations.

- Form $l := L|_{\mathfrak{g}} : \mathfrak{g} \to \mathbb{R}$

- Solve the Euler-Poincaré equations: $\frac{d}{dt}\frac{\delta l}{\delta \xi} = \pm \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}, \quad \xi(0) = \xi_0$

- Solve linear equation with time dependent coefficients (quadrature): $\dot{g}(t) = g(t)\xi(t)$ $(\dot{g}(t) = \xi(t)g(t)), \quad g(0) = e$

- Given $g_0 \in G$ the solution of the Euler-Lagrange equations is $V(t) = g_0 g(t)\xi(t)$ $(V(t) = \xi(t)g(t)g_0)$, initial condition $V(0) = g_0\xi_0$ $(V(0) = \xi_0 g_0)$.

Proof of Euler-Poincaré theorem

(i) and (ii) are equivalent by classical mechanics.

To show that (iii) and (iv) are equivalent, need to compute variations $\delta\xi(t) \in \mathfrak{g}$ induced on $\xi(t) = g(t)^{-1}\dot{g}(t) \in \mathfrak{g}$ by a variation $g(t,\lambda) \in G$ of $g(t)$, i.e., $g(t,\lambda)|_{\lambda=0} = g(t)$. Do the computation for left translation.

So, need to differentiate $g^{-1}(t,\lambda)\dot{g}(t,\lambda) \in \mathfrak{g}$ in the direction $\delta g(t) := \frac{d}{d\lambda}\Big|_{\lambda=0} g(t,\lambda) \in T_{g(t)}G$. Define $\eta(t) := g(t)^{-1}\delta g(t) \in \mathfrak{g}$. We have

$$\delta\xi = \frac{d}{d\lambda}\left(g^{-1}\frac{d}{dt}g\right)\bigg|_{\lambda=0} = -\left(g^{-1}\,\delta g\,g^{-1}\right)\dot{g} + g^{-1}\,\frac{\partial^2 g}{\partial\lambda\partial t}\bigg|_{\lambda=0}$$

$$\dot{\eta} = \frac{d}{dt}\left(g^{-1}\frac{d}{d\lambda}g\right)\bigg|_{\lambda=0} = -\left(g^{-1}\,\dot{g}\,g^{-1}\right)\delta g + g^{-1}\,\frac{\partial^2 g}{\partial t\partial\lambda}\bigg|_{\lambda=0}$$

$$\Longrightarrow \quad \delta\xi - \dot{\eta} = [\xi, \eta]$$

This proof is for matrix groups. Can be done totally abstractly.

Proof that (iii) and (iv) are equivalent:

$$\delta \int_{t_0}^{t_1} l(\xi) = \int_{t_0}^{t_1} \left\langle \frac{\delta l}{\delta \xi}, \delta\xi \right\rangle dt = \int_{t_0}^{t_1} \left\langle \frac{\delta l}{\delta \xi}, \dot\eta + \mathrm{ad}_\xi\, \eta \right\rangle dt$$
$$= \int_{t_0}^{t_1} \left\langle -\frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right) + \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}, \dot\eta \right\rangle dt$$

for all smooth curves $t \mapsto \eta(t)$ in $\mathfrak{g}$. This is equivalent to the Euler-Poincaré equations

$$\frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right) = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}$$

Comments on the proof: 1.) The constrained variations can deduced for any Lie group. G Lie group, $\mathfrak{g}$ its Lie algebra. Adopt the standard convention wherein the Lie bracket of $\mathfrak{g}$ is taken to be the Jacobi-Lie bracket of *left* invariant vector fields. The *left* (*right*) Maurer-Cartan forms $\Omega^1(G;\mathfrak{g})$ are

$$v_g \mapsto TL_g^{-1} \cdot v_g := g^{-1} v_g, \qquad v_g \in T_gG, \qquad left$$
$$v_g \mapsto TR_g^{-1} \cdot v_g := v_g g^{-1}, \qquad v_g \in T_gG, \qquad right.$$

Let $U \subset \mathbb{R}^2$ be an open set and let $g : U \to G$ be an embedding. Let (t, λ) be coordinates on U and $\partial\lambda$ and ∂_t the corresponding vector fields in $\mathfrak{X}(U)$. Define $\dot{g} := Tg \circ \partial_t$ and $\delta g := Tg \circ \partial_\lambda$. Finally, if $\xi := g^{-1}\dot{g}$ $(= \dot{g}g^{-1})$ and $\eta := g^{-1}\delta g$ $(= \delta g\, g^{-1})$, then

$$\partial_\lambda \xi = \partial_t \eta \pm [\xi, \eta]$$

2.) Concretely, $g(t, \lambda)$ appears as a deformation of a given smooth curve $g(t)$. These deformations are always assumed to be at least immersions, which are locally embeddings around a point in G.

3.) The usual bracket of vector fields is the *right* Lie algebra bracket, yet the formulas from Lie theory always use the *left* Lie algebra bracket. Of course $[\xi, \eta]_L = -[\xi, \eta]_R$. So, for the right Lie algebra, the above proposition states $\partial_s \xi = \partial_t \eta - [\eta, \xi]_R$.

4.) In continuum mechanicis applications, the group is Diff$_{(\text{vol})}(\mathcal{D})$ whose *left* Lie algebra is $\mathfrak{X}_{(\text{div},)||}(\mathcal{D})$ endowed with *minus* the standard Jacobi-Lie bracket of vector fields. Often one takes various subgroups of Diff$(\mathcal{D})$: volume preserving diffeomorphisms for incompressible hydrodynamics, symplectic diffeomorphisms for plasma physics, gauge groups for field theory, etc.

If $\varphi : U \subset \mathbb{R}^2 \to$ Diff$(\mathcal{D})$ is an embedding and $\dot{\varphi} = T\varphi \circ \partial_t$, $\delta\varphi = T\varphi \circ \partial_s$, $u(s,t) = \dot{\varphi}(s,t) \circ \varphi(s,t)^{-1}$, and $v(s,t) = \delta\varphi(s,t) \circ \varphi(s,t)^{-1}$, we have

$$\frac{\partial}{\partial s} u = \frac{\partial}{\partial t} v - [v, u]_{\text{Jacobi}-\text{Lie}}.$$

There is minus in front of the bracket precisely because there is a $+$ in the abstract theorem!

These are the Lin constraints (discovered by C.C. Lin in the 50s). We will come back to these constraints when studying various continuum mechanics examples later.

THE FREE TOP

The Lie algebra $\mathfrak{so}(3)$ and its dual

Underlying Lie group is the proper rotation group SO(3).

$(\mathfrak{so}(3), [\cdot, \cdot]) \cong (\mathbb{R}^3, \times)$ by the isomorphism

$$\mathbf{u} := (u^1, u^2, u^3) \in \mathbb{R}^3 \mapsto \hat{\mathbf{u}} := \begin{bmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{bmatrix} \in \mathfrak{so}(3).$$

Equivalently, this isomorphism is given by

$$\hat{\mathbf{u}}\mathbf{v} = \mathbf{u} \times \mathbf{v} \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^3.$$

Properties: for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$:

$$(\mathbf{u} \times \mathbf{v})\hat{} = [\hat{\mathbf{u}}, \hat{\mathbf{v}}] =: \mathrm{ad}_{\hat{\mathbf{u}}} \hat{\mathbf{v}}$$
$$[\hat{\mathbf{u}}, \hat{\mathbf{v}}]\mathbf{w} = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$
$$\mathbf{u} \cdot \mathbf{v} = -\frac{1}{2}\mathrm{trace}(\hat{\mathbf{u}}\hat{\mathbf{v}}).$$

If $A \in \mathrm{SO}(3)$, $\hat{\mathbf{u}} \in \mathfrak{so}(3)$ let $\mathrm{Ad}_A \hat{\mathbf{u}} := A\hat{\mathbf{u}}A^{-1}$. Then

$$(A\mathbf{u})\hat{} = \mathrm{Ad}_A \hat{\mathbf{u}} := A\hat{\mathbf{u}}A^T$$

$$A(\mathbf{u} \times \mathbf{v}) = A\mathbf{u} \times A\mathbf{v}$$

for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and $A \in \mathrm{SO}(3)$. This relation is *not* valid if A is just an orthogonal matrix; if A is not in the component of the identity matrix, then one gets a minus sign on the right hand side.

The dual $\mathfrak{so}(3)^*$ is identified with $\mathbb{R}^3$ by the isomorphism $\mathbf{\Pi} \in \mathbb{R}^3 \mapsto \widetilde{\mathbf{\Pi}} \in \mathfrak{so}(3)^*$ given by $\widetilde{\mathbf{\Pi}}(\hat{\mathbf{u}}) := \mathbf{\Pi} \cdot \mathbf{u}$ for any $\mathbf{u} \in \mathbb{R}^3$. Then the coadjoint action of $\mathrm{SO}(3)$ on $\mathfrak{so}(3)^*$ is given by

$$\mathrm{Ad}^*_{A^{-1}} \widetilde{\mathbf{\Pi}} = \widetilde{A\mathbf{\Pi}}.$$

The coadjoint action of $\mathfrak{so}(3)$ on $\mathfrak{so}(3)^*$ is given by

$$\mathrm{ad}^*_{\hat{\mathbf{u}}} \widetilde{\mathbf{\Pi}} = \widetilde{\mathbf{\Pi} \times \mathbf{u}}.$$

Continuum mechanical setup

Reference configuration: $\mathcal{B} \subset \mathbb{R}^3 = \{\mathbf{X} = (X^1, X^2, X^3)\}$; $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$

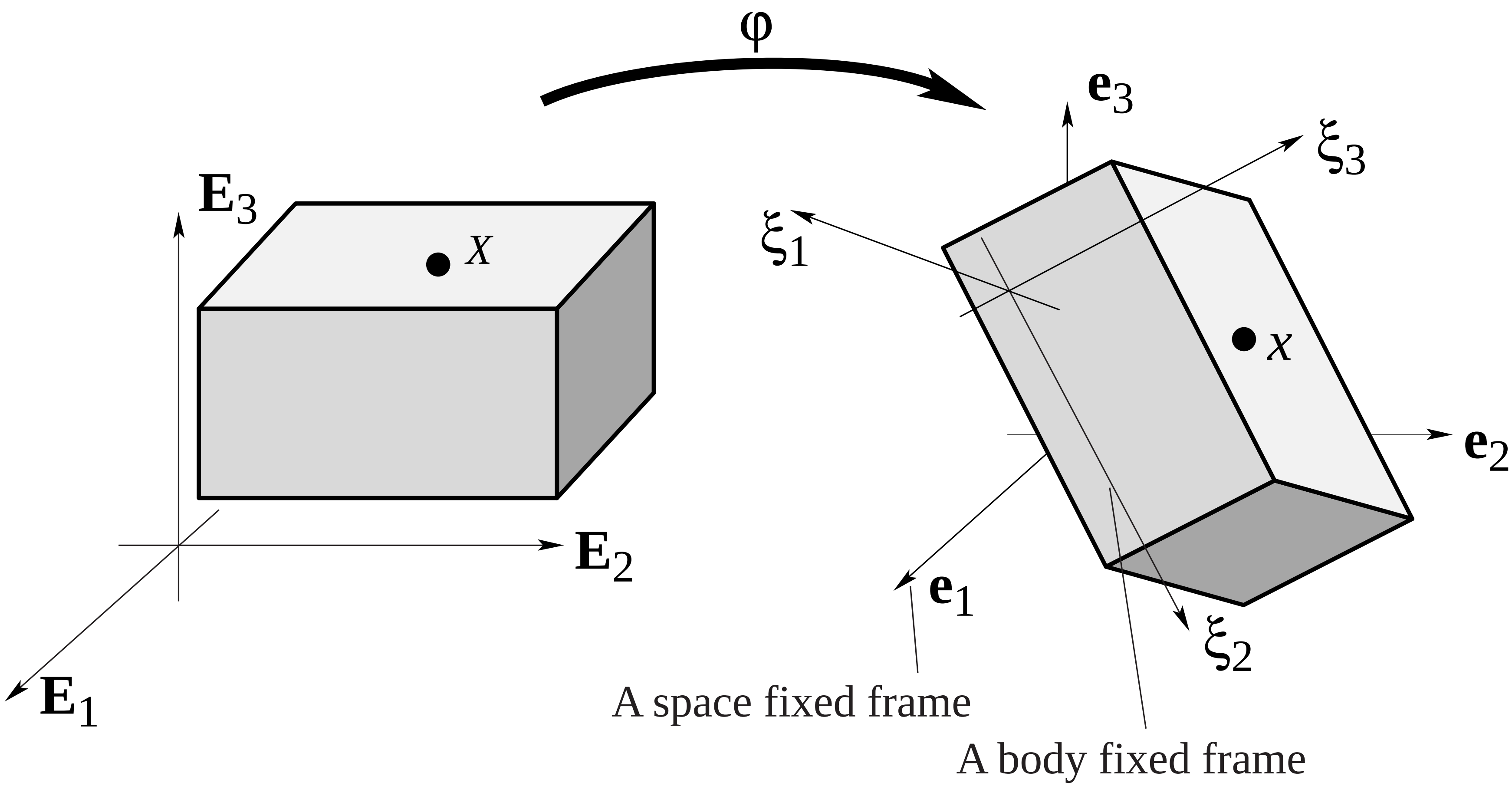
Spatial configuration: $\mathcal{S} = \mathbb{R}^3 = \{\mathbf{x} = (x^1, x^2, x^3)\}$; $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

Configuration: orientation preserving embedding $\mathcal{B} \to \mathcal{S}$

Motion: $\mathbf{x}(\mathbf{X}, t)$ time dependent family of configurations

For the rigid body moving about a fixed point, the motions are rotations: $\mathbf{x}(\mathbf{X}, t) := A(t)\mathbf{X}$, where $A(t) \in SO(3)$.

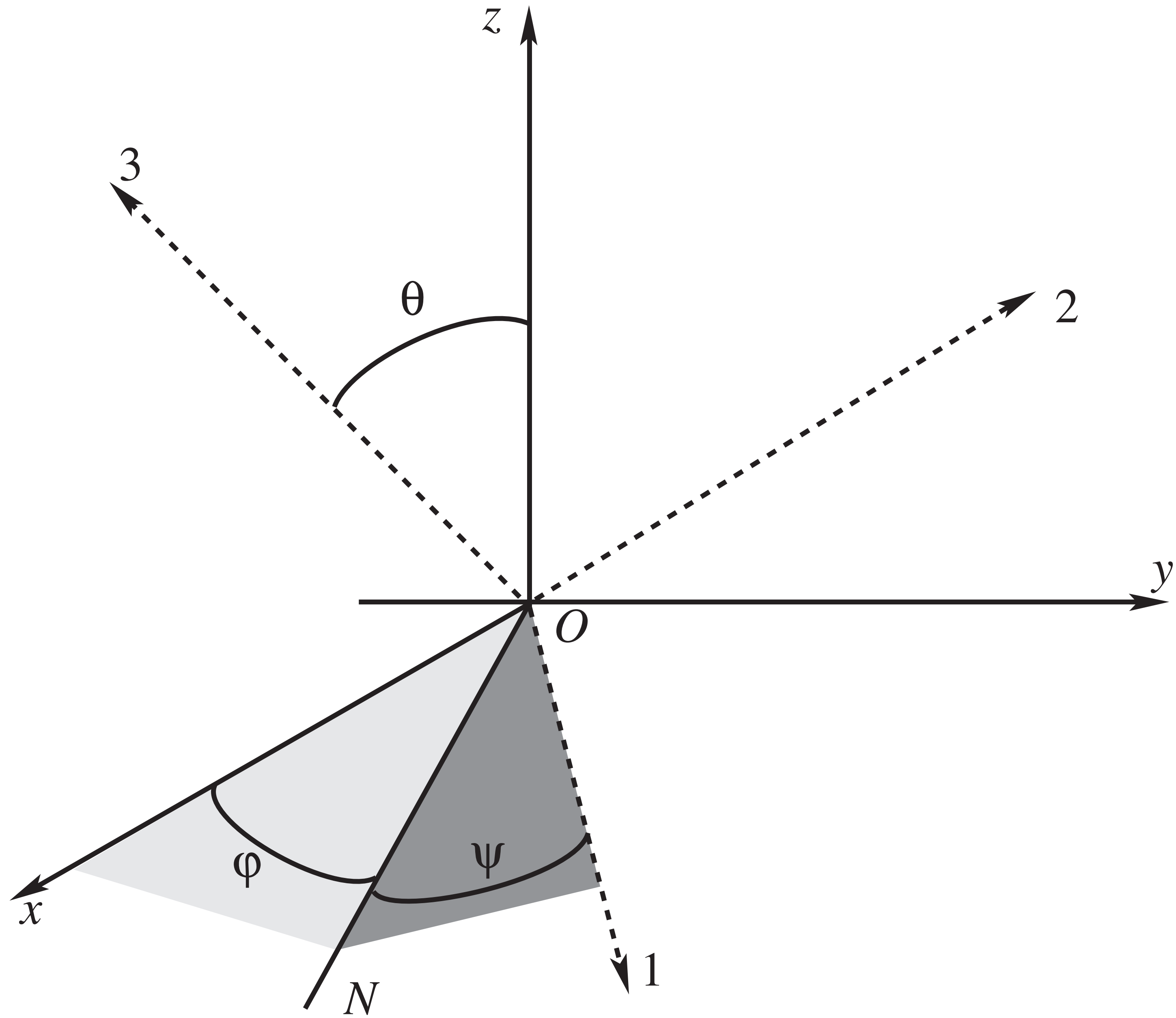
Time dependent orthonormal basis anchored in the body moving together with it: $\boldsymbol{\xi}_i := A(t)\mathbf{E}_i$, $i = 1, 2, 3$. ***Body*** or ***convected coordinates***: coordinates relative to $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$.


φ
$\mathbf{E}_3$
X
$\mathbf{E}_2$
$\mathbf{E}_1$
$\mathbf{e}_3$
ξ_3
ξ_1
x
$\mathbf{e}_2$
$\mathbf{e}_1$
ξ_2
A space fixed frame
A body fixed frame

Note that the components of a vector $\mathbf{V}$ relative to the basis $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ are the same as the components of the vector $A(t)\mathbf{V}$ relative to the basis $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$. In particular, the body coordinates of $\mathbf{x}(\mathbf{X}, t) = A(t)\mathbf{X}$ are X^1, X^2, X^3.

Euler angles: encode the passage from the spatial basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to the body basis $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$ by means of three consecutive counterclockwise rotations performed in a specific order: first rotate around the axis $\mathbf{e}_3$ by the angle φ and denote the resulting position of $\mathbf{e}_1$ by ON (line of nodes), then rotate about ON by the angle θ and denote the resulting position of $\mathbf{e}_3$ by $\boldsymbol{\xi}_3$, and finally rotate about $\boldsymbol{\xi}_3$ by the angle ψ.

By construction: $0 \leq \varphi, \psi < 2\pi$, $0 \leq \theta < \pi$. Bijection between (φ, ψ, θ) and SO(3). *Not* a chart since its differential vanishes at $\varphi = \psi = \theta = 0$. So for $0 < \varphi, \psi < 2\pi$, $0 < \theta < \pi$ the ***Euler angles*** (φ, ψ, θ) form a chart.


z
3
θ
2
y
O
φ
ψ
x
1
N

The resulting linear map performing the motion $\mathbf{x}(\mathbf{X}, t) = A(t)\mathbf{X}$ has the matrix relative to the bases $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ equal to

$$A = \begin{bmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\theta\sin\psi \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{bmatrix}$$

The ***material*** or ***Lagrangian velocity*** is defined by

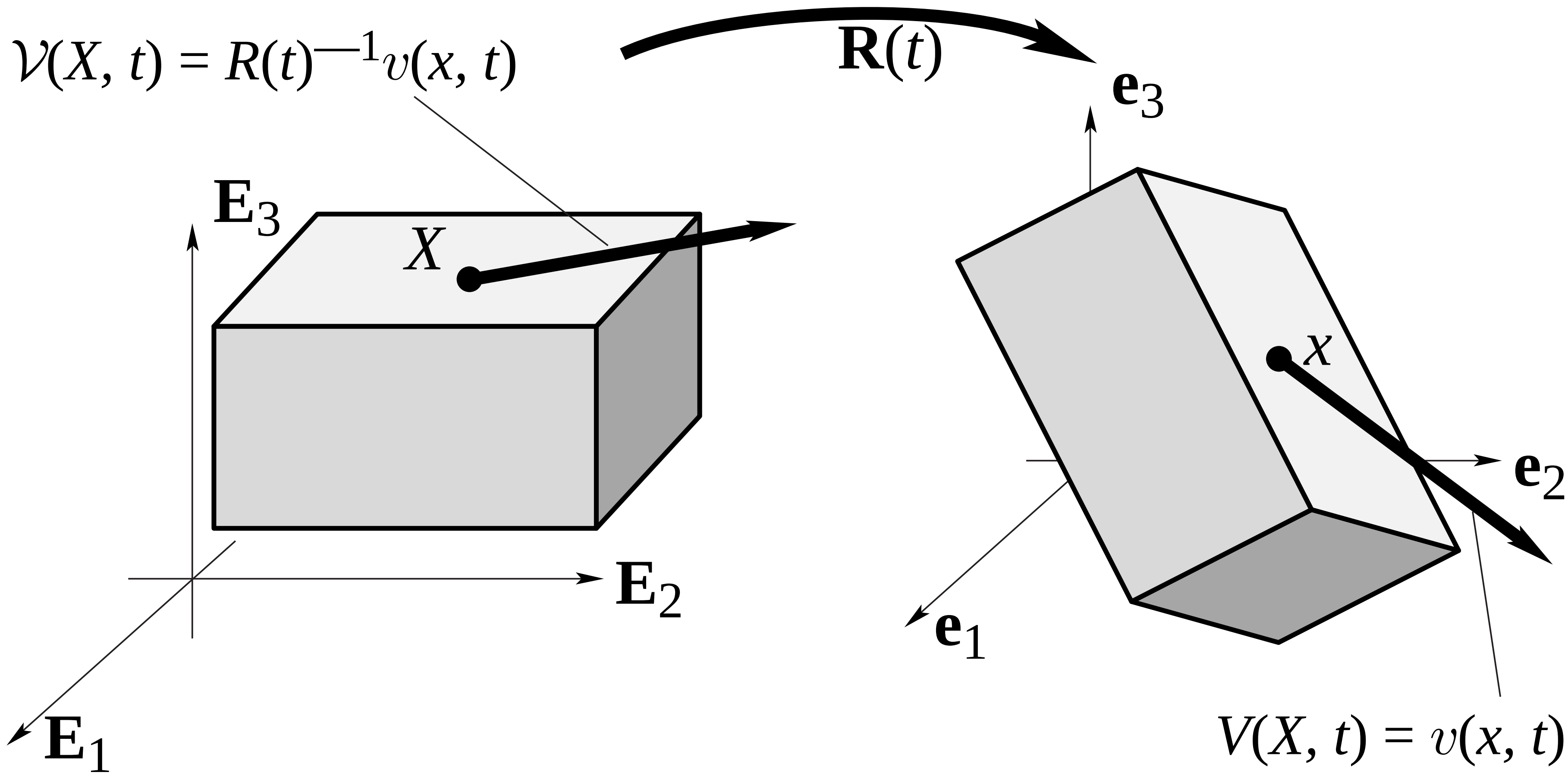
$$\mathbf{V}(\mathbf{X}, t) := \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t} = \dot{A}(t)\mathbf{X}.$$

The ***spatial*** or ***Eulerian velocity*** is defined by

$$\mathbf{v}(\mathbf{x}, t) := \mathbf{V}(\mathbf{X}, t) = \dot{A}(t)\mathbf{X} = \dot{A}(t)A(t)^{-1}\mathbf{x}.$$

The ***body*** or ***convective velocity*** is defined by

$$\begin{aligned} \mathcal{V}(\mathbf{X}, t) : &= -\frac{\partial \mathbf{X}(\mathbf{x}, t)}{\partial t} = A(t)^{-1}\dot{A}(t)A(t)^{-1}\mathbf{x} \\ &= A(t)^{-1}\mathbf{V}(\mathbf{X}, t) = A(t)^{-1}\mathbf{v}(\mathbf{x}, t). \end{aligned}$$


𝒱(X, t) = R(t)—1v(x, t)
R(t)
e3
E3
X
x
e2
E2
e1
E1
V(X, t) = v(x, t)

Kinetic energy

ρ_0 density in the reference configuration. The kinetic energy at time t in material, spatial, and convective representation:

$$\begin{aligned} K(t) &= \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{X}) \|\mathbf{V}(\mathbf{X},t)\|^2 d^3\mathbf{X} \\ &= \frac{1}{2}\int_{A(t)\mathcal{B}} \rho_0(A(t)^{-1}\mathbf{x}) \|\mathbf{v}(\mathbf{x},t)\|^2 d^3\mathbf{x} \\ &= \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{X}) \|\mathcal{V}(\mathbf{X},t)\|^2 d^3\mathbf{X}. \end{aligned}$$

Define $\quad \widehat{\boldsymbol{\omega}}(t) := \dot{A}(t)A(t)^{-1}, \qquad \widehat{\boldsymbol{\Omega}}(t) := A(t)^{-1}\dot{A}(t), \quad$ then

$$\mathbf{v}(\mathbf{x},t) = \boldsymbol{\omega}(t) \times \mathbf{x}, \qquad \mathcal{V}(\mathbf{X},t) = \boldsymbol{\Omega}(t) \times \mathbf{X},$$

so $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are the ***spatial*** and ***body angular velocities***. The expressions of $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ in Euler angles are

$$\boldsymbol{\omega} = \begin{bmatrix} \dot{\theta}\cos\varphi + \dot{\psi}\sin\varphi\sin\theta \\ \dot{\theta}\sin\varphi - \dot{\psi}\cos\varphi\sin\theta \\ \dot{\varphi} + \dot{\psi}\cos\theta \end{bmatrix} \quad \boldsymbol{\Omega} = \begin{bmatrix} \dot{\theta}\cos\psi + \dot{\varphi}\sin\psi\sin\theta \\ -\dot{\theta}\sin\psi + \dot{\varphi}\cos\psi\sin\theta \\ \dot{\varphi}\cos\theta + \dot{\psi} \end{bmatrix}.$$

So $$K(t) = \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{X}) \|\boldsymbol{\Omega}(t) \times \mathbf{X}\|^2 d^3\mathbf{X} =: \frac{1}{2}\langle\!\langle \boldsymbol{\Omega}(t), \boldsymbol{\Omega}(t) \rangle\!\rangle$$

which is the quadratic form of the bilinear symmetric map on $\mathbb{R}^3$

$$\langle\!\langle \mathbf{a}, \mathbf{b} \rangle\!\rangle := \int_{\mathcal{B}} \rho_0(\mathbf{X})(\mathbf{a} \times \mathbf{X}) \cdot (\mathbf{b} \times \mathbf{X}) d^3\mathbf{X} = \mathbb{I}\mathbf{a} \cdot \mathbf{b},$$

where $\mathbb{I} : \mathbb{R}^3 \to \mathbb{R}^3$ is the symmetric isomorphism (relative to the dot product) whose components are $\mathbb{I}_{ij} := \mathbb{I}\mathbf{E}_j \cdot \mathbf{E}_i = \langle\!\langle \mathbf{E}_j, \mathbf{E}_i \rangle\!\rangle$, i.e.,

$$\mathbb{I}_{ij} = -\int_{\mathcal{B}} \rho_0(\mathbf{X}) X^i X^j d^3\mathbf{X} \quad \text{if} \quad i \neq j$$
$$\mathbb{I}_{ii} = \int_{\mathcal{B}} \rho_0(\mathbf{X}) \left(\|\mathbf{X}\|^2 - (X^i)^2 \right) d^3\mathbf{X}.$$

So $\mathbb{I}$ is the ***moment of inertia tensor***. The basis in which it is diagonal is called the ***principal axis body frame*** and the diagonal elements I_1, I_2, I_3 of $\mathbb{I}$ in this basis are called the ***principal moments of inertia*** of the top. From now on, we choose the basis $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ to be a principal axis body frame. Hence the kinetic energy is

$$
\begin{aligned}
K(\boldsymbol{\Omega}) = \frac{1}{2}\boldsymbol{\Omega}\cdot\mathbb{I}\boldsymbol{\Omega} = -\frac{1}{4}\text{trace}\left(\widehat{\boldsymbol{\Omega}}(\widehat{\mathbb{I}\boldsymbol{\Omega}})\right) &= -\frac{1}{4}\text{trace}\left(\widehat{\boldsymbol{\Omega}}(\widehat{\boldsymbol{\Omega}}\Lambda + \Lambda\widehat{\boldsymbol{\Omega}})\right) \\
= \frac{1}{2}\Big[I_1(\dot{\varphi}\sin\psi\sin\theta + \dot{\theta}\cos\psi)^2 \\
+ I_2(\dot{\varphi}\cos\psi\sin\theta - \dot{\theta}\sin\psi)^2 + I_3(\dot{\varphi}\cos\theta + \dot{\psi})^2\Big]
\end{aligned}
$$

where $\Lambda = \text{diag}(\Lambda_1, \Lambda_2, \Lambda_3)$, $\Lambda_1 = (-I_1 + I_2 + I_3)/2$, $\Lambda_2 = (I_1 - I_2 + I_3)/2$, and $\Lambda_3 = (I_1 + I_2 - I_3)/2$, or $I_1 = \Lambda_2 + \Lambda_3$, $I_2 = \Lambda_3 + \Lambda_1$, and $I_3 = \Lambda_1 + \Lambda_2$. So, intrinsically, the kinetic energy on $TSO(3)$

$$
K(A, \dot{A}) = -\frac{1}{4}\text{trace}((\Lambda A^{-1}\dot{A} + A^{-1}\dot{A}\Lambda)A^{-1}\dot{A})
$$

is *left invariant* (action is $B\cdot(A,\dot{A}) := (BA, B\dot{A})$). It is the kinetic energy of the left invariant Riemannian metric on $SO(3)$ obtained by left translating the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$. So the solutions of the free rigid body motion project to geodesics on SO(3) relative to the left invariant metric whose value at the identity is $\langle\!\langle\cdot,\cdot\rangle\!\rangle$.

Identify $\langle\langle \mathbf{\Omega}, \cdot \rangle\rangle \in (\mathbb{R}^3)^*$ with the vector $\mathbf{\Pi} := \mathbb{I}\mathbf{\Omega} \in \mathbb{R}^3$, so

$$\mathbf{\Pi} = \begin{bmatrix} I_1(\dot{\varphi} \sin\psi \sin\theta + \dot{\theta}\cos\psi) \\ I_2(\dot{\varphi}\cos\psi \sin\theta - \dot{\theta}\sin\psi) \\ I_3(\dot{\varphi}\cos\theta + \dot{\psi}) \end{bmatrix}.$$

Define the conjugate variables by the Legendre transformation

$$p_\varphi := \frac{\partial K}{\partial \dot{\varphi}}, \quad p_\psi := \frac{\partial K}{\partial \dot{\psi}}, \quad p_\theta := \frac{\partial K}{\partial \dot{\theta}}, \quad \text{so}$$

$$\begin{aligned} K(\mathbf{\Pi}) &= \frac{1}{2}\mathbf{\Pi} \cdot \mathbb{I}^{-1}\mathbf{\Pi} = \frac{1}{2}\left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right) \\ &= \frac{1}{2}\left[\frac{[(p_\varphi - p_\psi\cos\theta)\sin\psi + p_\theta \sin\theta\cos\psi]^2}{I_1\sin^2\theta}\right. \\ &\quad \left. + \frac{[(p_\varphi - p_\psi\cos\theta)\cos\psi - p_\theta\sin\theta\sin\psi]^2}{I_2\sin^2\theta} + \frac{p_\psi^2}{I_3}\right]. \end{aligned}$$

The equations of motion

Chart on $T^*\,\mathrm{SO}(3)$, Euler angles and conjugate momenta

$$\dot\varphi = \frac{\partial K}{\partial p_\varphi}, \qquad \dot\psi = \frac{\partial K}{\partial p_\psi}, \qquad \dot\theta = \frac{\partial K}{\partial p_\theta}$$
$$\dot p_\varphi = -\frac{\partial K}{\partial \varphi}, \qquad \dot p_\psi = -\frac{\partial K}{\partial \psi}, \qquad \dot p_\theta = -\frac{\partial K}{\partial \theta}.$$

$$\mathbf{J}_R : (\varphi, \psi, \theta, p_\varphi, p_\psi, p_\theta) \mapsto \mathbf{\Pi}$$

A lengthy direct computation shows that these equations imply the ***Euler equations*** $\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbf{\Omega}$. Can be obtained in two ways.

(i) Canonical Poisson bracket of two functions $f, h : T^*SO(3) \to \mathbb{R}$ in a chart given by the Euler angles and their conjugate momenta

$$\{f,h\} = \frac{\partial f}{\partial \varphi}\frac{\partial h}{\partial p_\varphi} - \frac{\partial f}{\partial p_\varphi}\frac{\partial h}{\partial \varphi} + \frac{\partial f}{\partial \psi}\frac{\partial h}{\partial p_\psi} - \frac{\partial f}{\partial p_\psi}\frac{\partial h}{\partial \psi} + \frac{\partial f}{\partial \theta}\frac{\partial h}{\partial p_\theta} - \frac{\partial f}{\partial p_\theta}\frac{\partial h}{\partial \theta}.$$

A direct long computation shows that if $F, H : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, then

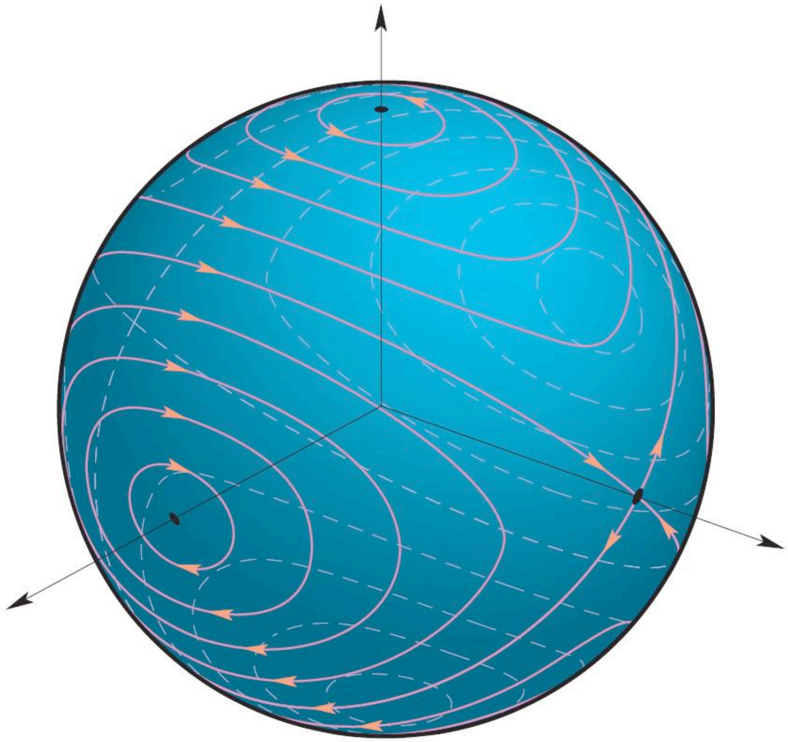
$$\{F \circ \mathbf{J}_R, H \circ \mathbf{J}_R\} = \{F, H\}_- \circ \mathbf{J}_R,$$ where

$$\{F,H\}_{-}(\mathbf{\Pi}) = -\mathbf{\Pi}\cdot(\nabla F\times\nabla H)$$

is the ***Lie-Poisson bracket*** on $\mathfrak{so}(3)^* \cong \mathbb{R}^3$. Of course, one can cite the Lie-Poisson reduction theorem avoiding all computations! The equation $\dot{F} = \{F,H\}$ for any $F : \mathbb{R}^3 \to \mathbb{R}$ is equivalent to the Euler equations obtained by Lie-Poisson reduction.

$\{F,\Phi(\|\mathbf{\Pi}\|^2)\} = 0, \forall F \in C^\infty(\mathbb{R}^3)$, so $\|\mathbf{\Pi}\|^2$ is the ***Casimir function*** of the Lie-Poisson structure on $\mathfrak{so}(3)^*$. So the body angular momentum $\mathbf{\Pi}$ evolves on concentric spheres. On the sphere of radius $\|\mathbf{\Pi}\|$, the Euler equation is Hamiltonian relative to the symplectic form $\omega_{-}(\mathbf{\Pi}) = -\frac{1}{\|\mathbf{\Pi}\|}da$, where $da(\mathbf{\Pi})\Big(\mathbf{u}\times\mathbf{\Pi},\mathbf{v}\times\mathbf{\Pi}\Big) = \|\mathbf{\Pi}\|\mathbf{\Pi}\cdot(\mathbf{u}\times\mathbf{v})$. The solutions of the Euler equation $\dot{\mathbf{\Pi}} = \mathbf{\Pi}\times\mathbf{\Omega}$ are therefore obtained by intersecting concentric spheres $\{\mathbf{\Pi} \mid \|\mathbf{\Pi}\| = R\}$ with the family of ellipsoids $\{\mathbf{\Pi} \mid \mathbf{\Pi}\cdot\mathbb{I}^{-1}\mathbf{\Pi} = C\}$ for any constants $R, C \geq 0$.

Stability Theorem: *There are six equilibria, four of them stable and two of them unstable. The stable ones correspond to rotations about the short and long axes of the moment of inertia and the unstable one corresponds to rotations about the middle axis.*



How do you solve the geodesic equations?

1.) Solve $\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbf{\Omega}$, $\mathbf{\Pi} = \mathbb{I}\mathbf{\Omega}$; done with Jacobi elliptic functions

2.) Solve $\dot{A} = A\widehat{\mathbf{\Omega}}$, $A(0) = I$; time-ordered integral; quadrature

3.) Solutions $t \mapsto A(t)$ are the geodesics on SO(3). $\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbf{\Omega}$, $\dot{A} = A\widehat{\mathbf{\Omega}}$ is the geodesic spray when $T\,\mathrm{SO}(3) \cong_{\text{left}} \mathrm{SO}(3) \times \mathbb{R}^3$.

(ii) Do Euler-Poincaré reduction by hand. Given is the Lagrangian $L(\mathbf{\Omega}) := K(\mathbf{\Omega}) = \frac{1}{2}\mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Omega}$ and consider the variational principle:

$$\delta \int_a^b L(\mathbf{\Omega}) dt = 0$$

but only subject to the restricted variations of the form

$$\delta\mathbf{\Omega} := \dot{\mathbf{\Sigma}} + \mathbf{\Omega} \times \mathbf{\Sigma}$$

where $\mathbf{\Sigma}(t) \in \mathbb{R}^3$ is arbitrary such that $\mathbf{\Sigma}(a) = \mathbf{\Sigma}(b) = 0$.

This is equivalent to the Euler equations. In fluids: Lin constraints.

Proof: From $\nabla L(\mathbf{\Omega}) = \mathbb{I}\mathbf{\Omega} = \mathbf{\Pi}$, we get

$$\begin{aligned} 0 &= \delta \int_a^b L(\mathbf{\Omega}) dt = \int_a^b \nabla L(\mathbf{\Omega}) \cdot \delta\mathbf{\Omega} dt = \int_a^b \mathbf{\Pi} \cdot \delta\mathbf{\Omega} dt \\ &= \int_a^b \mathbf{\Pi} \cdot (\dot{\mathbf{\Sigma}} + \mathbf{\Omega} \times \mathbf{\Sigma}) dt = -\int_a^b \dot{\mathbf{\Pi}} \cdot \mathbf{\Sigma} dt + \int_a^b \mathbf{\Pi} \cdot (\mathbf{\Omega} \times \mathbf{\Sigma}) dt \\ &= \int_a^b \left(-\dot{\mathbf{\Pi}} + \mathbf{\Pi} \times \mathbf{\Omega} \right) \cdot \mathbf{\Sigma} dt. \end{aligned}$$

The arbitrariness of $\mathbf{\Sigma}$ yields the Euler equations. ■

Left action induces a $\mathbb{R}^3$-valued conservation law. What is it?

Note that $A\boldsymbol{\Pi} = A\mathbb{I}\boldsymbol{\Omega} = (A\mathbb{I}A^{-1})(A\boldsymbol{\Omega}) = (A\mathbb{I}A^{-1})\boldsymbol{\omega}$ and since $A\mathbb{I}A^{-1} =: \mathbb{I}_{spat}$ is the moment of inertia in space, it follows that $\boldsymbol{\pi} := A\boldsymbol{\Pi}$ is the ***angular momentum in space***.

$$\begin{aligned}\dot{\boldsymbol{\pi}} &= \dot{A}\boldsymbol{\Pi} + A\dot{\boldsymbol{\Pi}} = (\dot{A}A^{-1})(A\boldsymbol{\Pi}) + A(\boldsymbol{\Pi} \times \boldsymbol{\Omega}) \\ &= \widehat{\boldsymbol{\omega}}\boldsymbol{\pi} + (A\boldsymbol{\Pi}) \times (A\boldsymbol{\Omega}) = \boldsymbol{\omega} \times \boldsymbol{\pi} + \boldsymbol{\pi} \times \boldsymbol{\omega} = \mathbf{0}\end{aligned}$$

so the angular momentum in space is conserved. Could have computed this directly as a momentum map. We will come back to this equation later – it has an important significance.

Note: The Euler top has too many conserved quantities. It is a non-commutatively integrable system and the motion takes place on 2-tori even though one would have expected it to be on 3-tori.

Note: $k = \frac{1}{2}\boldsymbol{\Omega} \cdot \mathbb{I}\boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}_{spat}\boldsymbol{\omega}$, so in space we have a NEW VARIABLE, the spatial moment of inertia tensor! The theory developed so far does not apply to this – come back later.

Historically, $\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbf{\Omega}$ was not understood: not canonical Hamiltonian because the equations are in $\mathbb{R}^3$. Not Lagrangian because it is first order. Yet, it comes from a classical system. Try to write it as a classical Hamiltonian system on a vector space.

Natural action of SU(2) on $\mathbb{C}^2$. Since this action is by isometries of the Hermitian inner product, it is automatically symplectic since the symplectic form is minus the imaginary part of the inner product. Hence, the equivariant momentum map $\mathbf{J} : \mathbb{C}^2 \to \mathfrak{su}(2)^*$ is

$$\langle \mathbf{J}(z, w), \xi \rangle = \frac{1}{2}\omega(\xi(z, w)^{\mathsf{T}}, (z, w)), \quad z, w \in \mathbb{C}, \quad \xi \in \mathfrak{su}(2).$$

$\mathfrak{su}(2)$ consists of 2×2 skew Hermitian matrices of trace zero. $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3)$ and hence to $(\mathbb{R}^3, \times)$ by

$$\mathbf{x} = (x^1, x^2, x^3) \in \mathbb{R}^3 \overset{\sim}{\longmapsto}$$

$$\tilde{\mathbf{x}} := \frac{1}{2}\begin{bmatrix} -ix^3 & -ix^1 - x^2 \\ -ix^1 + x^2 & ix^3 \end{bmatrix} \in \mathfrak{su}(2);$$

$$[\tilde{\mathbf{x}}, \tilde{\mathbf{y}}] = (\mathbf{x} \times \mathbf{y})\tilde{}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^3.$$

Other useful formulas are

$$\det(2\widetilde{\mathbf{x}}) = \|\mathbf{x}\|^2 \quad \text{and} \quad \operatorname{trace}(\widetilde{\mathbf{x}}\widetilde{\mathbf{y}}) = -\frac{1}{2}\mathbf{x}\cdot\mathbf{y}.$$

$\mathfrak{su}(2)^* \cong \mathbb{R}^3$ by the map $\mu \in \mathfrak{su}(2)^* \mapsto \breve{\mu} \in \mathbb{R}^3$ defined by

$$\breve{\mu}\cdot\mathbf{x} := -2\langle \mu, \widetilde{\mathbf{x}}\rangle, \quad \forall \mathbf{x} \in \mathbb{R}^3.$$

So $\breve{\mathbf{J}} : \mathbb{C}^2 \to \mathbb{R}^3$ is given by: for any $\mathbf{x} \in \mathbb{R}^3$ we have

$$\begin{aligned}
\breve{\mathbf{J}}(z,w)\cdot\mathbf{x} &= -2\langle \mathbf{J}(z,w), \widetilde{\mathbf{x}}\rangle \\
&= \frac{1}{2}\operatorname{Im}\left(\begin{bmatrix} -ix^3 & -ix^1 - x^2 \\ -ix^1 + x^2 & ix^3 \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix}\cdot\begin{bmatrix} z \\ w \end{bmatrix}\right) \\
&= -\frac{1}{2}(2\operatorname{Re}(w\overline{z}), 2\operatorname{Im}(w\overline{z}), |z|^2 - |w|^2)\cdot\mathbf{x}, \quad \text{so} \\
\breve{\mathbf{J}}(z,w) &= -\frac{1}{2}(2w\overline{z}, |z|^2 - |w|^2) \in \mathbb{R}^3.
\end{aligned}$$

The momentum map $\breve{\mathbf{J}} : (\mathbb{C}^2, -\operatorname{Im}\langle\!\langle\, ,\rangle\!\rangle) \to \mathbb{R}^3_+$ is Poisson and hence $-\breve{\mathbf{J}} : (\mathbb{C}^2, -\operatorname{Im}\langle\!\langle\, ,\rangle\!\rangle) \to \mathbb{R}^3_-$ is Poisson.

Hamilton's equations in $(z, w) \in \mathbb{C}^2$ for $H \circ (-\check{\mathbf{J}}) : \mathbb{C}^2 \to \mathbb{R}$ (**collective Hamiltonian**) push forward by $-\check{\mathbf{J}}$ to $\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbb{I}^{-1}\mathbf{\Pi}$. (z, w) are the ***Cayley-Klein parameters***. They represent a first attempt to understand the rigid body equations as a Hamiltonian system, before the introduction of Poisson manifolds. In quantum mechanics, the same variables are called the ***Kustaanheimo-Stiefel coordinates***. Similar construction in fluid dynamics: ***Clebsch variables*** for the Euler equations.

Try to understand the map $-\check{\mathbf{J}}(z, w) = \frac{1}{2}(2w\bar{z}, |z|^2 - |w|^2)$ better.

If $(z, w) \in S^3 := \left\{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\right\}$, then $\| - \check{\mathbf{J}}(z, w)\| = 1/2$, so that $-\check{\mathbf{J}}|_{S^3} : S^3 \to S^2_{1/2} \subset \mathbb{R}^3$ sphere of radius 1/2.

$-\check{\mathbf{J}}|_{S^3}$ is surjective and its fibers are circles. Indeed, given $(x^1, x^2, x^3) = (x^1 + ix^2, x^3) = (re^{i\psi}, x^3) \in S^2_{1/2}$,

$$-\check{\mathbf{J}}^{-1}(re^{i\psi}, x^3) = \left\{ \left(e^{i\theta}\sqrt{\frac{1}{2} + x^3}, e^{i\varphi}\sqrt{\frac{1}{2} - x^3} \right) \in S^3 \,\middle|\, e^{i(\theta - \varphi + \psi)} = 1 \right\}.$$

so $-\check{\mathbf{J}}|_{S^3} : S^3 \to S^2_{1/2}$ is the ***Hopf fibration***. Hence:

The momentum map of the SU(2)-action on $\mathbb{C}^2$, the Cayley-Klein parameters, the Kustaanheimo-Stiefel coordinates, and the family of Hopf fibrations on concentric three-spheres in $\mathbb{C}^2$ are ***the same map****.*

For interesting applications, we need a vastly enlarged version of the Euler-Poincaré reduction theorem. We state it only. The proof, follows the pattern given above.

AFFINE EULER-POINCARÉ REDUCTION

Right G-representation on V, $(v,g) \in V \times G \mapsto vg \in V$, induces:

- right G-representation on V^*: $(a,g) \in V^* \times G \mapsto ag \in V^*$
- right $\mathfrak{g}$-representation on V: $(v,\xi) \in V \times \mathfrak{g} \mapsto v\xi \in V$
- right $\mathfrak{g}$-representation on V^*: $(a,\xi) \in V^* \times \mathfrak{g} \mapsto a\xi \in V^*$

Duality pairings: $\langle\,,\rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle\,,\rangle_V : V^* \times V \to \mathbb{R}$
Recall $-\langle a\xi, v\rangle_V = \langle a, v\xi\rangle_V$

Affine *right* representation: $\theta_g(a) = ag + c(g)$, where $c \in \mathcal{F}(G, V^*)$ is a right group one-cocycle, i.e., $c(fg) = c(f)g + c(g)$, $\forall f, g \in G$. This implies that $c(e) = 0$ and $c(g^{-1}) = -c(g)g^{-1}$. Note that

$$\left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(t\xi)}(a) = a\xi + \mathbf{d}c(\xi), \quad \xi \in \mathfrak{g}, \quad a \in V^*,$$

where $\mathbf{d}c : \mathfrak{g} \to V^*$ is defined by $\mathbf{d}c(\xi) := T_e c(\xi)$. Useful to introduce:

- $\mathbf{d}c^{\mathsf{T}} : V \to \mathfrak{g}^*$ by $\langle \mathbf{d}c^{\mathsf{T}}(v), \xi\rangle_{\mathfrak{g}} := \langle \mathbf{d}c(\xi), v\rangle_V$, for $\xi \in \mathfrak{g}$, $v \in V$

- $\diamond : V \times V^* \to \mathfrak{g}^*$ by $\langle v \diamond a, \xi\rangle_{\mathfrak{g}} := -\langle a\xi, v\rangle_V$, for $\xi \in \mathfrak{g}$, $v \in V$, $a \in V^*$

- then: $\langle a\xi + \mathbf{d}c(\xi), v\rangle_V = \langle \mathbf{d}c^{\mathsf{T}}(v) - v \diamond a, \xi\rangle_{\mathfrak{g}}$

- the semidirect product $S = G \circledS V$ with group multiplication

$$(g_1, v_1)(g_2, v_2) := (g_1 g_2, v_2 + v_1 g_2), \quad g_i \in G, \quad v_i \in V$$

- its Lie algebra $\mathfrak{s} = \mathfrak{g} \circledS V$ with bracket

$$\mathrm{ad}_{(\xi_1, v_1)}(\xi_2, v_2) := [(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], v_1\xi_2 - v_2\xi_1)$$

- then for $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^* = \mathfrak{g}^* \times V^*$ we have

$$\mathrm{ad}^*_{(\xi, v)}(\mu, a) = (\mathrm{ad}^*_\xi \mu + v \diamond a, a\xi)$$

In a physical problem (like liquid crystals) we are given:

• $L : TG \times V^* \to \mathbb{R}$ right G-invariant under the action $(v_h, a) \in T_hG \times V^* \stackrel{g}{\longmapsto} (v_h g, \theta_g(a)) = (v_h g, ag + c(g)) \in T_{hg}G \times V^*$.

• So, if $a_0 \in V^*$, define $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then L_{a_0} is right invariant under the lift to TG of right translation of $G^c_{a_0}$ on G, where $G^c_{a_0}$ is the θ-isotropy group of a_0.

• Right G-invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l(v_g g^{-1}, \theta_{g^{-1}}(a_0)) = L(v_g, a_0).$$

• Curve $g(t) \in G$, let $\xi(t) := \dot{g}(t)g(t)^{-1} \in \mathfrak{g}$, $a(t) = \theta_{g(t)^{-1}}(a_0) \in V^*$ Then $a(t)$ as the unique solution of the following affine differential equation with time dependent coefficients and initial condition $a(0) = a_0 \in V^*$

$$\dot{a}(t) = -a(t)\xi(t) - \mathbf{d}c(\xi(t)),$$

The following are equivalent:

(i) With a_0 held fixed, Hamilton's variational principle

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0,$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.
(ii) $g(t)$ satisfies the Euler-Lagrange equations for L_{a_0} on G.
(iii) The constrained variational principle

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0,$$

holds on $\mathfrak{g} \times V^*$, upon using variations of the form

$$\delta\xi = \frac{d\eta}{dt} - [\xi, \eta], \quad \delta a = -a\eta - \mathbf{d}c(\eta),$$

for all smooth curves $t \mapsto \eta(t) \in \mathfrak{g}$ vanishes at the endpoints.
(iv) The affine Euler-Poincaré equations hold on $\mathfrak{g} \times V^*$:

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = -\operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a - \mathbf{d}c^{\mathsf{T}}\left(\frac{\delta l}{\delta a}\right).$$

Example: Euler top

Want to write equations of motion in spatial representation, i.e., using spatial angular momentum $\boldsymbol{\omega} = \dot{A}A^{-1}$, one has to implement reduction relative to the right translation by SO(3). Such a symmetry can be obtained only by introducing a NEW VARIABLE, the spatial inertial tensor $\mathbb{I}_{spat} = A\mathbb{I}A^{-1}$. In this case, the associated reduction process yields the Lagrangian

$$k = \ell_{spat}(\boldsymbol{\omega}, \mathbb{I}_{spat}) = \frac{1}{2}\mathbb{I}_{spat}\boldsymbol{\omega}\cdot\boldsymbol{\omega} = -\frac{1}{4}\mathrm{trace}((\Lambda A^{-1}\dot{A} + A^{-1}\dot{A}\Lambda)A^{-1}\dot{A}).$$

Euler-Poincaré: variations $\delta\boldsymbol{\omega} = \dot{\boldsymbol{\varphi}} - \boldsymbol{\omega}\times\boldsymbol{\varphi}$, $\delta\mathbf{I}_{spat} = [\boldsymbol{\varphi}, \mathbf{I}_{spat}]$, where $\boldsymbol{\varphi}$ is a arbitrary curve in $\mathbb{R}^3$ vanishing at the endpoints,

$$\delta\int_{t_0}^{t_1}\ell(\boldsymbol{\omega}, \mathbf{I}_{spat})dt = 0,$$

is equivalent to (recall $\boldsymbol{\pi} = \mathbf{I}_{spat}\boldsymbol{\omega}$)

$$\frac{d}{dt}\boldsymbol{\pi} = \frac{d}{dt}(\mathbf{I}_{spat}\boldsymbol{\omega}) = 0, \qquad \frac{d}{dt}\mathbf{I}_{spat} = [\omega, \mathbf{I}_{spat}].$$

The first equation is the conservation of the angular momentum in space. We have seen this before. $3+6=9$ equations.

THE HEAVY TOP

Same kinetic energy.

The potential energy U is determined by the height of the center of mass over the horizontal plane in the spatial representation.

- ℓ length of segment from fixed point to center of mass
- $\boldsymbol{\chi}$ unit vector from origin on this segment
- $M = \int_{\mathcal{B}} \rho_0(\mathbf{X}) d^3\mathbf{X}$ total mass of the body
- g magnitude of gravitational acceleration
- $\boldsymbol{\Gamma}(t) := Mg\ell A(t)^{-1}\mathbf{e}_3$, spatial Oz unit vector viewed in body description
- $\boldsymbol{\lambda}(t) := Mg\ell A(t)\boldsymbol{\chi}$, unit vector on the line connecting the origin with the center of mass viewed in the spatial description

$$
\begin{aligned}
U &= Mg\ell\mathbf{e}_3 \cdot A(t)\boldsymbol{\chi} && \textit{material/Lagrangian} \\
&= \mathbf{e}_3 \cdot \boldsymbol{\lambda} && \textit{spatial/Eulerian} \\
&= \boldsymbol{\Gamma} \cdot \boldsymbol{\chi} && \textit{body/convective}
\end{aligned}
$$

New complications appear: **There are new variables, depending on the representation; λ in the spatial and Γ in the body representation**

QUESTION: $L = K - U : T^*SO(3) \to \mathbb{R}^3$. The parameters are $\mathbf{e}_3, \boldsymbol{\chi} \in \mathbb{R}^3$, $\mathbb{I} \in \mathrm{Sym}_2$, $Mg\ell \in \mathbb{R}$. So dual of representation space: $V^* = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{Sym}_2$. Since the particles do not have a separate dynamics, there is no cocycle, so $c = 0$.

$$L(A, \dot{A}, \mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) = \frac{1}{2}\left(\mathbb{I}A^{-1}\dot{A}\right) \cdot \left(A^{-1}\dot{A}\right) - Mg\ell\mathbf{e}_3 \cdot A\boldsymbol{\chi} \qquad \textit{material}$$

Heavy top in body representation. Left SO(3)***-representation***: $B \cdot (\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) := (B\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi})$, $\forall B \in \mathrm{SO}(3)$. Since

$$L(BA, B\dot{A}, B\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) = L(A, \dot{A}, \mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}), \qquad \forall B \in \mathrm{SO}(3),$$

general theory says that we have Euler-Poincaré equations and associated variational principles for the **body Lagrangian**

$$L_B(\boldsymbol{\Omega}, \boldsymbol{\Gamma}, \mathbb{I}, \boldsymbol{\chi}) := L(I, A^{-1}\dot{A}, A^{-1}\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) = \frac{1}{2}\boldsymbol{\Omega} \cdot \mathbb{I}\boldsymbol{\Omega} - \boldsymbol{\Gamma} \cdot \boldsymbol{\chi} \qquad \textit{body}$$

Since $\frac{\delta L_B}{\delta \boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} = \boldsymbol{\Pi}$ and $\frac{\delta L_B}{\delta \boldsymbol{\Gamma}} = -\boldsymbol{\chi}$, the abstract Euler-Poincaré equations become the standard **Euler-Poisson equations**

$$\dot{\boldsymbol{\Pi}} = \boldsymbol{\Pi} \times \boldsymbol{\Omega} + \boldsymbol{\Gamma} \times \boldsymbol{\chi}, \quad \dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma} \times \boldsymbol{\Omega}, \quad \dot{\mathbb{I}} = 0, \quad \dot{\boldsymbol{\chi}} = 0.$$

Heavy top in spatial representation. ***Right*** SO(3)-***representation***: $(\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) \cdot B := (\mathbf{e}_3, B^{-1}\mathbb{I}B, B^{-1}\boldsymbol{\chi})$, $\forall B \in$ SO(3). Since

$$L(AB, \dot{A}B, \mathbf{e}_3, B^{-1}\mathbb{I}B, B^{-1}\boldsymbol{\chi}) = L(A, \dot{A}, \mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}), \qquad \forall B \in \mathrm{SO}(3),$$

general theory says that we have Euler-Poincaré equations and associated variational principles for the **spatial Lagrangian**

$$L_S(\boldsymbol{\omega}, \mathbf{e}_3, \mathbb{I}_S, \boldsymbol{\lambda}) := L(I, \dot{A}A^{-1}, \mathbf{e}_3, A\mathbb{I}A^{-1}, A\boldsymbol{\chi}) = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}_{spat}\boldsymbol{\omega} - \mathbf{e}_3 \cdot \boldsymbol{\lambda}$$

Since $\frac{\delta L_S}{\delta \boldsymbol{\omega}} = \mathbb{I}_{spat}\boldsymbol{\omega} = \boldsymbol{\pi}$, $\frac{\delta L_S}{\delta \boldsymbol{\lambda}} = -\mathbf{e}_3$, $\frac{\delta L_S}{\delta \mathbb{I}_{spat}} = \boldsymbol{\omega} \otimes \boldsymbol{\omega}$, we get

$$\dot{\boldsymbol{\pi}} = \mathbf{e}_3 \times \boldsymbol{\lambda}, \quad \dot{\mathbf{e}}_3 = 0, \quad \dot{\mathbb{I}}_{spat} = \left[\mathbb{I}_{spat}, \widehat{\boldsymbol{\omega}}\right], \quad \dot{\boldsymbol{\lambda}} = \boldsymbol{\omega} \times \boldsymbol{\lambda}$$

Remark: In body representation, we have equations on $\mathfrak{se}(3)^* = \mathbb{R}^3 \times \mathbb{R}^3$. Four dimensional generic orbits; Casimirs are $\mathbf{\Pi} \cdot \mathbf{\Gamma}$, $\|\mathbf{\Gamma}\|^2$.

In spatial representation, equations are on the dual of the semidirect product $\mathfrak{so}(3) Ⓢ (\mathrm{Sym}^2 \times \mathbb{R}^3)$. This is 12 dimensional. It has 6 Casimirs: the three invariants of $\mathbb{I}_{spat}$, $\|\boldsymbol{\lambda}\|^2$, $(\mathbb{I}_{spat}\boldsymbol{\lambda}) \cdot \boldsymbol{\lambda}$, $\|\mathbb{I}_{spat}\boldsymbol{\lambda}\|^2$. The generic coadjoint orbit is symplectomorphic to $(T^*\,\mathrm{SO}(3), can)$. One more integral: $\boldsymbol{\pi} \cdot \mathbf{e}_3$. Reduce and get to 4 dimensions $(TS^2, magnetic)$.

Remark: There is a Hamiltonian version of this theorem, the **semidirect product reduction theorem with cocycles**. It produces a Poisson bracket, symplectic leaves which are orbits of the coadjoint action augmented by a cocycle, explicit expression of the symplectic form on these orbits, Hamilton's equations.

Before doing fluids and elasticity, let's recall the standard continuum mechanics setup.

CONTINUUM MECHANICS SETTING

Reference configuration: $(\mathcal{B}, \mathbf{G})$ oriented Riemannian manifold
Usually $\mathcal{B} \subset \mathbb{R}^3 = \{\mathbf{X} = (X^1, X^2, X^3)\}$; $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ orthonormal

Spatial configuration: $(\mathcal{S}, \mathbf{g})$ oriented Riemannian manifold
Usually $\mathcal{S} = \mathbb{R}^3 = \{\mathbf{x} = (x^1, x^2, x^3)\}$; $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ orthonormal

Configuration: orientation preserving embedding $\varphi : \mathcal{B} \to \mathcal{S}$, so the ***configuration space*** is $\mathrm{Emb}_+(\mathcal{B}, \mathcal{S})$

Motion: $\varphi_t(\mathbf{X}) = \mathbf{x}(\mathbf{X}, t)$ time dependent family of configurations

Time dependent basis anchored in the body moving together with it: $\boldsymbol{\xi}_i := \varphi_t(\mathbf{E}_i)$, $i = 1, 2, 3$. ***Body*** or ***convected coordinates***: coordinates relative to $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$.

The ***material*** or ***Lagrangian velocity*** is defined by

$$\mathbf{V}(\mathbf{X},t) := \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} = \frac{\partial}{\partial t}\varphi_t(\mathbf{X}).$$

The ***spatial*** or ***Eulerian velocity*** is defined by

$$\mathbf{v}(\mathbf{x},t) := \mathbf{V}(\mathbf{X},t) \Longleftrightarrow \mathbf{v}_t \circ \varphi_t = \mathbf{V}_t.$$

The ***body*** or ***convective velocity*** is defined by

$$\mathcal{V}(\mathbf{X},t) := -\frac{\partial \mathbf{X}(\mathbf{x},t)}{\partial t} = -\frac{\partial}{\partial t}\varphi_t^{-1}(\mathbf{x}) \Longleftrightarrow \mathcal{V}_t = T\varphi_t^{-1} \circ \mathbf{V}_t = \varphi_t^* \mathbf{v}_t$$

The ***particle relabeling group*** $\mathrm{Diff}(\mathcal{B})$ acts on the *right* on $\mathrm{Emb}_+(\mathcal{B},\mathcal{S})$. The ***material frame indifference group*** $\mathrm{Diff}(\mathcal{S})$ acts on the *left* on $\mathrm{Emb}_+(\mathcal{B},\mathcal{S})$.

In continuum mechanics it is important to keep all options open and always have three descriptions available. They serve different purposes and the interactions between them gives interesting physical insight.

FIXED BOUNDARY BAROTROPIC FLUIDS

$\mathcal{D} := \mathcal{B} = \mathcal{S}$ oriented Riemannian manifold, $\partial\mathcal{D}$ smooth, $\mathbf{G} = \mathbf{g} =: g$, $G = \mathrm{Diff}(\mathcal{D})$, $\mathfrak{g} = \mathfrak{X}(\mathcal{D})$, $V = S^2(\mathcal{D})$, $V^* = S_2(\mathcal{D}) \otimes |\Omega^n(\mathcal{D})|$

$$L_{(\bar{\varrho},g)}(V_\eta) = \frac{1}{2}\int_{\mathcal{D}} g(\eta(X))(V_\eta(X), V_\eta(X))\bar{\varrho}(X) - \int_{\mathcal{D}} E(\bar{\varrho}(\_), g(\eta(\_)), T\_\eta)(X)\bar{\varrho}(X), \qquad \textit{material}$$

$$\ell_{spat}(\mathbf{v}, \bar{\rho}, g) = \frac{1}{2}\int_{\mathcal{D}} g(x)(\mathbf{v}(x), \mathbf{v}(x))\bar{\rho}(x) - \int_{\mathcal{D}} e(\rho)(x)\bar{\rho}(x), \qquad \textit{spatial}$$

$$\ell_{conv}(\mathcal{V}, \bar{\varrho}, C) = \frac{1}{2}\int_{\mathcal{D}} C(X)(\mathcal{V}(X), \mathcal{V}(X))\bar{\varrho}(X) - \int_{\mathcal{D}} \mathcal{E}(\bar{\varrho}, C)(X)\bar{\varrho}(X), \; \textit{body}$$

- $\bar{\varrho}(X) =: \varrho(X)\mu(g)(X) := (\eta^*\bar{\rho})(X)$, $\quad \bar{\rho}(x) := \rho(x)\mu(g)(x)$ mass density
- $C := \eta^* g$ Cauchy-Green tensor
- $E(\bar{\varrho}(\_), g(\eta(\_)), T\_\eta) := e\left(\frac{\bar{\varrho}}{\mu(\eta^* g)}\right) = e(\rho) \circ \eta$, $\mathcal{E}(\bar{\varrho}, C) := e\left(\frac{\bar{\varrho}}{\mu(C)}\right) = e(\rho) \circ \eta$ internal energy density

L is *right*-invariant under the action of $\varphi \in \text{Diff}(\mathcal{D})$ given by

$$(V_\eta, \bar{\varrho}, g) \mapsto (V_\eta \circ \varphi, \varphi^* \bar{\varrho}, g)$$

and the reduction map

$$(V_\eta, \bar{\varrho}, g) \mapsto (\mathbf{v}, \bar{\rho}, g) := (V_\eta \circ \eta^{-1}, \eta_* \bar{\varrho}, g)$$

induces the spatial Lagrangian $\ell_{spat}(\mathbf{v}, \rho, g)$ because

$$E(\bar{\varrho}, g \circ \eta, T\eta) \mapsto E(\varphi^* \bar{\varrho}, g \circ \eta \circ \varphi, T\eta \circ T\varphi) = E(\bar{\varrho}, g \circ \eta, T\eta) \circ \varphi$$

when $(\eta, \bar{\varrho}) \mapsto (\eta \circ \varphi, \varphi^* \bar{\varrho})$. g is not acted on by $\text{Diff}(\mathcal{D})$.
L is *left*-invariant under the action of $\psi \in \text{Diff}(\mathcal{D})$ given by

$$(V_\eta, \bar{\varrho}, g) \mapsto (T\psi \circ V_\eta, \bar{\varrho}, \psi_* g).$$

and the reduction map

$$(V_\eta, \bar{\varrho}, g) \mapsto (\mathcal{V}, \bar{\varrho}, C) := (T\eta^{-1} \circ V_\eta, \bar{\varrho}, \eta^* g),$$

induces the convective Lagrangian $\ell_{conv}(\mathcal{V}, \bar{\varrho}, C)$ because

$$E(\bar{\varrho}, g \circ \eta, T\eta) \mapsto E(\bar{\varrho}, \psi_* g \circ (\psi \circ \eta), T\psi \circ T\eta) = E(\bar{\varrho}, g \circ \eta, T\eta)$$

when $(\eta, g) \mapsto (\psi \circ \eta, \psi_* g)$. $\bar{\varrho}$ is not acted on by $\text{Diff}(\mathcal{D})$.

General semidirect product reduction gives spatial equations

$$\begin{cases} \partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v} = -\frac{1}{\rho} \operatorname{grad}_g p, \quad p = \rho^2 \frac{\partial e}{\partial \rho} \\ \partial_t \rho + \operatorname{div}_g(\rho \mathbf{v}) = 0, \quad \mathbf{v} \| \partial \mathcal{D}, \end{cases}$$

and convective equations

$$\begin{cases} \bar{\varrho}\,(\partial_t \mathcal{V} + \nabla_{\mathcal{V}} \mathcal{V}) = 2 \operatorname{Div}_C \left(\frac{\partial \mathcal{E}}{\partial C} \bar{\varrho} \right) \\ \partial_t C - \boldsymbol{\pounds}_{\mathcal{V}} C = 0, \quad \mathcal{V} \| \partial \mathcal{B}, \end{cases}$$

right hand side is related to the spatial pressure p by the formula

$$2 \frac{\partial \mathcal{E}}{\partial C} \bar{\varrho} = -(p \circ \eta) \mu(C) C^\sharp, \quad \text{so} \quad 2 \operatorname{Div}_C \left(\frac{\partial \mathcal{E}}{\partial C} \bar{\varrho} \right) = - \operatorname{grad}_C (p \circ \eta) \mu(C),$$

$C^\sharp \in S^2(\mathcal{D})$ is the cometric, grad_C is the gradient relative to C.

Important special case: ideal homogeneous incompressible fluid. Group is $\operatorname{Diff}_{\mu(g)}(\mathcal{D}) := \{\eta \in \operatorname{Diff}(\mathcal{D}) \mid \eta^* \mu(g) = \mu(g)\}$, dual of representation space is $V^* = S_2(\mathcal{D})$.

Lagrangian in spatial and convective rep. (suppose $H^1(\mathcal{D}, \mathbb{R}) = 0$):

So, this is the study of L^2-geodesics in spatial representation.

$$\ell_{spat}(\mathbf{v}, g) = \frac{1}{2}\int_{\mathcal{D}} g(x)(\mathbf{v}(x), \mathbf{v}(x))\mu(g)(x)$$
$$\ell_{conv}(\mathcal{V}, \bar{\varrho}, C) = \frac{1}{2}\int_{\mathcal{D}} C(X)(\mathcal{V}(X), \mathcal{V}(X))\mu(g)(X)$$

In spatial representation: if $\mathfrak{X}_{div,\|}(\mathcal{D})^* = \mathfrak{X}_{div,\|}(\mathcal{D}) \Longrightarrow$

$$\partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v} = -\operatorname{grad} p \qquad \textit{Euler equations}$$

if $\mathfrak{X}_{div,\|}(\mathcal{D})^* = \mathbf{d}\Omega^1_{\delta,\|}(\mathcal{D}) := \{\mathbf{d}\mathbf{v}^{\flat_g} \mid \mathbf{v} \in \mathfrak{X}_{div,\|}(\mathcal{D})\} = \Omega^2_{ex}(\mathcal{D}) \Longrightarrow$

$$\partial_t \omega + \pounds_{\mathbf{v}} \omega = 0, \text{ where } \omega := \mathbf{d}\mathbf{v}^{\flat_g} \text{ vorticity} \qquad \textit{vorticity advection}$$

In convective representation: if $\mathfrak{X}_{div,\|}(\mathcal{D})^* = \Omega^1_{\delta,\|}(\mathcal{D}) \Longrightarrow$

$$\partial_t \mathbb{P}\left(\mathcal{V}^{\flat_C}\right) = 0 \quad \text{and} \quad \partial_t C - \pounds_{\mathcal{V}} C = 0.$$

$\mathbb{P} : \Omega^1(\mathcal{D}) \to \Omega^1_{\delta,\|}(\mathcal{D})$ orthogonal Hodge projector for the metric g

if $\mathfrak{X}_{div,\|}(\mathcal{D})^* = \Omega^2_{ex}(\mathcal{D}) \Longrightarrow$

$$\partial_t \Omega = 0 \quad \text{and} \quad \partial_t C - \pounds_{\mathcal{V}} C = 0.$$

where $\Omega := \mathbf{d}\mathcal{V}^{\flat_C}$ is the convective vorticity.

Go back to spatial representation.

The geodesic $t \mapsto \eta_t \in \text{Diff}_{\mu(g)}(\mathcal{D})$ is given by solving the equation $\partial\eta_t/\partial t = \mathbf{v}_t \circ \eta_t$ with the velocity $t \mapsto \mathbf{v}_t$ found after solving

- either the Euler equations for $\mathbf{v}$
- or the vorticity advection equation for ω and then inverting the relation $\omega = \mathbf{d}\mathbf{v}^{\flat_g}$ with boundary condition $g(\mathbf{v}, \mathbf{n}) = 0$ on $\partial\mathcal{D}$.

Coadjoint action of $\eta \in \text{Diff}_{\mu(g)}(\mathcal{D})$ on $\omega \in \mathbf{d}\Omega^1(\mathcal{D})$: $\text{Ad}^*_{\eta^{-1}}\omega = \eta_*\omega$.

Coadjoint orbit: $\mathcal{O}_\omega = \left\{\eta_*\omega \mid \eta \in \text{Diff}_{\mu(g)}(\mathcal{D})\right\}$, i.e., all smooth rearrangements of initial ω.

Coadjoint action of $\mathfrak{X}_{\text{div},\parallel}(M)$ on $\mathbf{d}\Omega^1(M) \cong \mathfrak{X}_{\text{div},\parallel}(M)^*$ is hence given by $\text{ad}^*_v\omega = \pounds_v\omega$.

The following statements are true and equivalent to each other:

(i) *The vorticity ω is transported by the flow η_t of $\mathbf{v}$.* If $\omega_0 = \omega_{t=0}$,

$$\left.\frac{d}{dt}\right|_{t=0} (\eta_t)_* \omega_0 = -(\eta_t)_* \boldsymbol{\pounds}_{\mathbf{v}} \omega_0 = -\boldsymbol{\pounds}_{\mathbf{v}} (\eta_t)_* \omega_0,$$

so $\omega_t = (\eta_t)_* \omega_0$ is the only solution with ω_0 as initial condition.

(ii) *Solution curves of the vorticity advection equation remain on coadjoint orbits in $\mathfrak{X}^*_{\mathrm{div},\|}(\mathcal{D})$.* Indeed, the solution is $\omega = (\eta_t)_* \omega_0$, where η_t is the flow of $\mathbf{v}$.

(iii) *Kelvin's circulation theorem: For any loop C in $\mathcal{D}$ bounding a surface S, the circulation*

$$\oint_{C_t} \mathbf{v}^{\flat_g} = \text{constant},$$

where $C_t := \eta_t(C)$ and η_t is the flow of $\mathbf{v}$. Indeed, by change of variables and Stokes' theorem, for $S_t := \eta_t(S)$, we have

$$\oint_{C_t} \mathbf{v}^{\flat_g} = \iint_{S_t} \mathbf{d}\mathbf{v}^{\flat_g} = \iint_{S_t} \omega = \iint_{S_t} (\eta_t)_* \omega_0 = \iint_S \omega_0 = \text{constant}.$$

ELASTICITY

Euler-Poincaré theory does not apply; do by hand with EP as guide.
BC: Displacement (η given on part of $\partial\mathcal{B}$); traction ($\mathbf{P}\cdot\mathbf{N}_C|_{\partial\mathcal{B}} = \tilde{\boldsymbol{\tau}}$).
Configuration space Emb($\mathcal{B},\mathcal{S}$). Material Lagrangian:

$$L(V_\eta, \bar{\varrho}, g, G) = \frac{1}{2}\int_{\mathcal{B}} g(\eta(X))(V_\eta(X), V_\eta(X))\bar{\varrho}(X) - \int_{\mathcal{B}} W(g(\eta(\_)), T\_\eta, G(\_))(X)\bar{\varrho}(X).$$

Material frame indifference: the *material stored energy function* W is invariant under the transformations

$$(\eta, g) \mapsto (\psi\circ\eta, \psi_* g), \quad \psi \in \text{Diff}(\mathcal{S}), \qquad i.e.,$$

$$W\left(\psi_* g(\psi(\eta(\_))), T_{\eta(\_)}\psi\circ T\_\eta, G(\_)\right) = W\left(g(\eta(\_)), T\_\eta, G(\_)\right).$$

$\forall\eta \in$ Emb($\mathcal{B},\mathcal{S}$), $\forall\psi : \eta(\mathcal{B}) \to \eta(\mathcal{B})$ diffeomorphism

So can define the **convective stored energy** $\mathcal{W}$ by

$$\mathcal{W}(C(X), G(X)) := W\left(\eta^* g(X), \mathbf{I}, G(X)\right) = W\left(g(\eta(X)), T_X\eta, G(X)\right).$$

Convective quantities: $C := \eta^* g$ Cauchy-Green tensor,

$(\mathcal{V}, \bar{\varrho}, C, G) := \left(T\eta^{-1} \circ V_\eta, \bar{\varrho}, \eta^* g, G\right) \in \mathfrak{X}(\mathcal{B}) \times |\Omega^n(\mathcal{B})| \times S_2(\mathcal{B}) \times S_2(\mathcal{B})$,

$$\ell_{conv}(\mathcal{V}, \varrho, C, G) = \frac{1}{2}\int_{\mathcal{B}} C(\mathcal{V}, \mathcal{V})\varrho - \int_{\mathcal{B}} \mathcal{W}(C, G)\varrho.$$

Convective equations of motion:

$$\varrho\left(\partial_t \mathcal{V} + \nabla_{\mathcal{V}} \mathcal{V}\right) = 2\,\mathrm{Div}_C\left(\frac{\partial \mathcal{W}}{\partial C}\varrho\right), \qquad \partial_t C - \pounds_{\mathcal{V}} C = 0.$$

So, elasticity has always a convective representation. Spatial rep.?

Isotropy: Need invariance under the *right* action of $\mathrm{Diff}(\mathcal{B})$:

$$(V_\eta, \varrho, g, G) \mapsto (V_\eta \circ \varphi, \varphi^* \varrho, g, \varphi^* G), \quad \varphi \in \mathrm{Diff}(\mathcal{B})$$

Kinetic energy is right-invariant. So sufficient condition is

$W\Big(g(\eta(\varphi(X))), T_X(\eta\circ\varphi), \varphi^* G(X)\Big) = \Big(W(g(\eta(\_)), T_\_\eta, G(\_))\circ\varphi\Big)(X)$,

for all $\varphi \in \mathrm{Diff}(\mathcal{B})$. This is equivalent to

$$\mathcal{W}(\varphi^* C, \varphi^* G) = \mathcal{W}(C, G) \circ \varphi, \qquad \forall \varphi \in \mathrm{Diff}(\mathcal{B})$$

This is ***material covariance*** which implies isotropy.

Spatial quantities: $c := \eta_* G \in \mathcal{S}_2(D_\Sigma)$ **Finger deformation tensor**

$$\mathbf{u} := \dot{\eta} \circ \eta^{-1} \in \mathfrak{X}(D_\Sigma), \quad \bar{\rho} := \eta_* \bar{\varrho} \in |\Omega^n(D_\Sigma)|,$$

$\Sigma = \eta(\partial \mathcal{B})$ boundary of *current configuration* $D_\Sigma := \eta(\mathcal{B}) \subset \mathcal{S}$,

$$w_\Sigma(c, g) := \mathcal{W}(\eta^* g, \eta^* c) \circ \eta^{-1}$$

spatial stored energy function. w_Σ, $\mathcal{W}$, and W are related by

$$(w_\Sigma(c, g) \circ \eta)(X) = \mathcal{W}(\eta^* g(X), \eta^* c(X)) = W\left(g(\eta(X)), T_X \eta, \eta^* c(X)\right).$$

Doyle-Ericksen formula for the Cauchy stress tensor

$$\boldsymbol{\sigma} = 2\rho \frac{\partial w_\Sigma}{\partial g} \in S^2(D_\Sigma)$$

Force per unit of deformed area with normal $\mathbf{n}$ is $\mathbf{t}(x, t, \mathbf{n})$, the Cauchy traction vector. In $\mathbb{R}^3$, if balance of momentum holds: $\forall \mathcal{U} \subset \mathcal{B}$ open,

$$\frac{d}{dt} \int_{\varphi_t(\mathcal{U})} \rho \mathbf{u} d^3 x = \int_{\varphi_t(\mathcal{U})} \rho \mathbf{b} d^3 x + \int_{\partial \varphi_t(\mathcal{U})} \mathbf{t} da(x)$$

where $\mathbf{t}$ is evaluated on the outward unit normal to $\partial \varphi_t(\mathcal{U})$, then $\mathbf{t}(x, t, \mathbf{n}) = \boldsymbol{\sigma}(x, t) \cdot \mathbf{n}(x, t)$. $\mathbf{b}$ given external body force per unit mass.

In $\mathbb{R}^3$, if balance of moment of momentum holds: $\forall \mathcal{U} \subset \mathcal{B}$ open,

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho(x \times \mathbf{u}) d^3x = \int_{\varphi_t(\mathcal{U})} \rho(x \times \mathbf{b}) d^3x + \int_{\partial\varphi_t(\mathcal{U})} (x \times (\boldsymbol{\gamma} \cdot \mathbf{n})) da(x)$$

where $\mathbf{t}$ is evaluated on the outward unit normal to $\partial\varphi_t(\mathcal{U})$, then $\boldsymbol{\sigma}$ is symmetric.

Reduced Lagrangian

$$\ell_{spat}(\Sigma, \mathbf{v}, \bar{\rho}, g, c) = \frac{1}{2}\int_{D_\Sigma} g(\mathbf{v}, \mathbf{v})\bar{\rho} - \int_{D_\Sigma} w_\Sigma\,(c, g)\,\bar{\rho},$$

variables defined on current configuration D_Σ and Σ is a variable.

Spatial equations of motion: (BC) $\mathbf{v}|_{\Sigma_d} = 0, \quad \boldsymbol{\sigma} \cdot \mathbf{n}_g|_{T\Sigma_\tau} = 0$

$$\rho\,(\partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v}) = \mathrm{Div}_g\,(\boldsymbol{\sigma})\,, \qquad \partial_t c + \boldsymbol{\pounds}_{\mathbf{v}} c = 0, \qquad \partial_t \bar{\rho} + \boldsymbol{\pounds}_{\mathbf{v}} \bar{\rho} = 0,$$

$$\partial_t \Sigma = g(\mathbf{v}, \mathbf{n}_g)$$

SYMMETRIC REPRESENTATION OF THE RIGID BODY EQUATIONS

A.) **The n-dimensional free rigid body**

Invariant inner product on $\mathfrak{so}(n)$

$$\langle \xi, \eta \rangle := -\frac{1}{2}\operatorname{trace}(\xi\eta),$$

$\xi, \eta \in \mathfrak{so}(n)$, identifies $\mathfrak{so}(n)$ with $\mathfrak{so}(n)^*$.

The **left invariant n-dimensional free rigid body equations** are

$$\dot{Q} = Q\Omega \quad \text{and} \quad \dot{M} = [M, \Omega], \qquad \text{(RBn)}$$

$Q \in SO(n)$ is the attitude of the body, $\Omega := Q^{-1}\dot{Q}$ is the body angular velocity, $M := J(\Omega) = \Lambda\Omega + \Omega\Lambda \in \mathfrak{so}(n)$ is the body angular momentum. $J : \mathfrak{so}(n) \to \mathfrak{so}(n)$ is $\langle \cdot, \cdot \rangle$-symmetric, positive definite, $\Lambda = \operatorname{diag}(\Lambda_1, \ldots, \Lambda_n)$, $\Lambda_i + \Lambda_j > 0$ for all $i \neq j$.

The equations $\dot{M} = [M, \Omega]$ are the Euler-Poincaré equations on $\mathfrak{so}(n)$ for the Lagrangian $l(\Omega) = \frac{1}{2}\langle \Omega, J(\Omega)\rangle$. This corresponds to the Lagrangian on $T\,\mathrm{SO}(n)$ given by

$$L(g, \dot{g}) = \frac{1}{2}\left\langle g^{-1}\dot{g}, J(g^{-1}\dot{g})\right\rangle .$$

General Euler-Poincaré theory implies that $\dot{M} = [M, \Omega]$ are the second component of the geodesic equations on $T\,\mathrm{SO}(n)$, left trivialized as $\mathrm{SO}(n) \times \mathfrak{so}(n)$, relative to the left invariant metric whose expression at the identity is $\langle\langle \Omega_1, \Omega_2 \rangle\rangle = \langle \Omega_1, J(\Omega_2)\rangle$.

$\dot{M} = [M, \Omega]$ are integrable (Manakov [1976], Mishchenko-Fomenko [1976-1978]). Idea:

$$\dot{M} = [M, \Omega] \Longleftrightarrow \frac{d}{dt}(M + \lambda\Lambda^2) = [M + \lambda\Lambda^2, \Omega + \lambda\Lambda], \quad \text{so}$$

$$\frac{1}{k}\mathrm{trace}(M + \lambda\Lambda^2)^k = \sum_{i=0}^{k} p_i(M)\lambda^i$$

is conserved. Need to count correctly the $p_i(M)$, show involution, independence; $\frac{1}{2}\dim \mathcal{O}$ polynomials, $\mathcal{O}$ generic $SO(n)$ adjoint orbit.

B.) **Left symmetric representation**

$$\dot{Q} = Q\Omega; \qquad \dot{P} = P\Omega \tag{SRBn}$$

where Ω is regarded as a function of Q and P via the equations

$$\Omega := J^{-1}(M) \in \mathfrak{so}(n) \quad \text{and} \quad M := Q^T P - P^T Q.$$

It is easy to check that this system of equations on the space $\mathrm{SO}(n) \times \mathrm{SO}(n)$ is invariant under the left diagonal action of $\mathrm{SO}(n)$.

If (Q, P) is a solution of (SRBn), then (Q, M) where $M = J(\Omega)$ and $\Omega = Q^{-1}\dot{Q}$ satisfies the rigid body equations (RBn).

Proof: Differentiating $M = Q^T P - P^T Q$ and using the equations (SRBn) gives the second of the equations (RBn). ■

The spatial angular momentum (the momentum map for the cotangent lifted action of $SO(n)$ on $T^*SO(n)$) equals $m = PQ^T - QP^T$ and is conserved. More: PQ^T and QP^T are separately conserved.

C.) Local equivalence

Conversely, let $t \in \mathbb{R} \mapsto (Q, M) \in SO(n) \times \mathfrak{so}(n)$ solution of (RBn). To find a solution of (SRBn) need to solve for $P \in SO(n)$ in

$$M = Q^T P - P^T Q.$$

Since $\|Q\| = \|P\| = 1$ ($\|\cdot\|$ operator norm) $\Rightarrow \|M\| \leq 2$. Define $\mathcal{C} := \{(Q, P) \mid \|M\| = 2\}$, $\mathcal{S} := \{(Q, P) \mid \|M\| < 2\}$. Since $M \mapsto \|M\|$ is a Casimir and $\|\cdot\|$ is invariant under conjugation $\Longrightarrow$ $\mathcal{C}, \mathcal{S}$ are invariant under the flow of (SRBn).

Since, for any matrix A,

$$\sinh A := (e^A - e^{-A})/2 = A + \frac{1}{3!}A^3 + \frac{1}{5!}A^5 + \cdots$$

it follows that $\sinh : \mathfrak{so}(n) \to \mathfrak{so}(n)$. If $x \in \mathbb{R}$, $|x| < 1$, the series expansion of $\sinh^{-1} x$ is

$$\sinh^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}$$

So there is an inverse $\sinh^{-1} : \{A \in \mathfrak{so}(n) \mid \|A\| < 1\} \to \mathfrak{so}(n)$.

For $\|M\| < 2$, the equation $M = Q^T P - P^T Q$ has the solution $P = Q\left(e^{\sinh^{-1} M/2}\right)$.

Example $SO(3)$: $\sqrt{\frac{1}{2}\text{trace}\left(\hat{\mathbf{x}}\hat{\mathbf{x}}^T\right)} = \|\hat{\mathbf{x}}\| = \|\mathbf{x}\|$ for any $\mathbf{x} \in \mathbb{R}^3$

$\mathfrak{so}(3) = \{\mu\hat{\mathbf{x}} \mid \|\mathbf{x}\| = 1, \mu \geq 0\} \Longrightarrow \{A \in \mathfrak{so}(3) \mid \|A\| < 1\} = \{\mu\hat{\mathbf{x}} \mid \|\mathbf{x}\| = 1, 0 \leq \mu < 1\}$. Rodrigues' formula

$$e^{\mu\hat{\mathbf{x}}} = I + \sin(\mu)\hat{\mathbf{x}} + \left(I - \mathbf{x}\mathbf{x}^T\right)(\cos\mu - 1)$$

implies that $\sinh(\mu\hat{\mathbf{x}}) = \sin(\mu)\hat{\mathbf{x}}$ so $\sinh : \mathfrak{so}(3) \to \mathfrak{so}(3)$ is not globally one-to-one but it has an inverse $\sinh^{-1} : \{\|A\| < 1\} \to \mathfrak{so}(3)$ given by

$$\sinh^{-1}(\mu\hat{c}) = \sin^{-1}(\mu)\hat{c}.$$

We determine $\mathfrak{C} = \{(Q, P) \mid \|M\| = 2\}$. Since the exponential map is onto, $Q^T P = e^{\mu\hat{\mathbf{x}}} \Rightarrow M = Q^T P - (Q^T P)^T = 2\sinh(\mu\hat{\mathbf{x}}) = \sin(\mu)\hat{\mathbf{x}}$, so $\|M\| = 2 \Longleftrightarrow |\sin(\mu)| = 1 \Longleftrightarrow \mu = \pi/2 \Longleftrightarrow Q^T P = I + \hat{\mathbf{x}}$. Thus

$$\mathfrak{C} = \{(Q, P) \in SO(3) \times SO(3) \mid Q^T P = I + \hat{\mathbf{x}}, \|\mathbf{x}\| = 1\}$$

D.) Hamiltonian structure

Symplectic vector space $\mathfrak{gl}(n) \times \mathfrak{gl}(n)$ with symplectic form

$$\Omega_{\mathfrak{gl}(n)}((\xi_1, \eta_1), (\xi_2, \eta_2)) = \frac{1}{2}\operatorname{trace}(\eta_2^T \xi_1 - \eta_1^T \xi_2), \quad (\xi_i, \eta_i) \in \mathfrak{gl}(n) \times \mathfrak{gl}(n)$$

The Hamiltonian system given by

$$H(\xi, \eta) = -\frac{1}{8}\operatorname{trace}\left[\left(J^{-1}(\xi^T \eta - \eta^T \xi)\right)\left(\xi^T \eta - \eta^T \xi\right)\right].$$

leaves $SO(n) \times SO(n)$ invariant and induces on it (SRBn). $\mathcal{S}$ is a symplectic submanifold of $\mathfrak{gl}(n) \times \mathfrak{gl}(n)$. Poisson bracket at (Q, P):

$$\{F|_{\mathcal{S}}, K|_{\mathcal{S}}\} = \langle \nabla_2 K, \nabla_1 F \rangle - \langle \nabla_1 K, \nabla_2 F \rangle$$
$$- \frac{1}{2}\left\langle Q\,(\nabla_2 K)^T + (\nabla_2 K)\,Q^T, \left(I + R \otimes R^T\right)^{-1} R\left(P\,(\nabla_1 F)^T + (\nabla_1 F)\,P^T\right)\right\rangle$$
$$+ \frac{1}{2}\left\langle P\,(\nabla_1 K)^T + (\nabla_1 K)\,P^T, \left(I + R \otimes R^T\right)^{-1}\left(Q\,(\nabla_2 F)^T + (\nabla_2 F)\,Q^T\right) R\right\rangle,$$

$R = QP^T$, $F, K \in C^\infty(\mathfrak{gl}(n) \times \mathfrak{gl}(n))$ and ∇_1, ∇_2 partial gradients.

Why did this work? What is the general setup?

CLEBSCH OPTIMAL CONTROL

A.) (SRBn) as an optimal control problem

$T > 0$, $Q_0, Q_T \in \mathrm{SO}(n)$. **Rigid body optimal control problem**:

$$\min_{U \in \mathfrak{so}(n)} \frac{1}{4} \int_0^T \langle U, J(U) \rangle dt.$$

Constraint on U: there is a curve $Q(t) \in \mathrm{SO}(n)$ such that

$$\dot{Q} = QU \qquad Q(0) = Q_0, \qquad Q(T) = Q_T.$$

The rigid body optimal control problem has optimal evolution equations (SRBn) where P is the costate vector given by the Pontryagin Maximum Principle. The optimal control in this case is given by

$$U = J^{-1}(Q^T P - P^T Q).$$

Idea of proof: Apply Pontryagin Maximum Principle

$$\delta \int_0^T \left[\left\langle P, QU - \dot{Q} \right\rangle - \frac{1}{4} \langle U, J(U) \rangle \right] dt = 0.$$

The costate vector P is a multiplier enforcing the dynamics.

There are no constraints on the costate vector $P \in \mathfrak{gl}(n)$; can consider the restriction of the extremal flows to invariant submanifolds. This limits possible extremal trajectories that can be recovered. For example (SRBn) restricts to a system on $\mathrm{SO}(n) \times \mathrm{SO}(n)$. One can make other assumptions on the costate vector. For example, suppose we assume a costate vector B such that $Q^T B$ is skew. Then it is easy to check that that the extremal evolution equations become

$$\dot{Q} = QJ^{-1}(Q^T B), \qquad \dot{B} = BJ^{-1}(Q^T B)$$

and that these equations restrict to an invariant submanifold defined by the condition that $Q^T B$ is skew symmetric. These are the McLachlan-Scovel equations [1995]. Comparing these equations with (SRBn) we see that $B = P - QP^T Q$.

One can discretize all of this and give algorithms.

QUESTION: What is the meaning of this strange formulation of the free rigid body equation?

IDEA: The optimal control problem should be the link that ties these equations to the Euler-Poincaré formulation.

B.) **Optimal control**

Q manifold, U vector space, $g : Q \times U \to \mathbb{R}$ **cost function**,
$X : Q \times U \to TQ$ smooth, $X_u := X(\cdot, u) \in \mathfrak{X}(Q)$.
Given $q_0, q_T \in Q$, consider the typical optimal control problem:

Find the curves $q = q(t) \in Q$, $u = u(t) \in U$ that minimize

$$\int_0^T g(q(t), u(t))dt$$

subject to the following conditions:
(A) $\dot{q}(t) = X(q(t), u(t))$;
(B) $q(0) = q_0$ and $q(T) = q_T$.

C.) **Pontryagin Maximum Principle**

Pontryagin function $\widehat{H} : T^*Q \times U \to \mathbb{R}$ for optimal control problem:

$$\widehat{H}(\alpha_q, u) := \langle \alpha_q, X(q, u) \rangle - p_0 g(q, u),$$

where $p_0 \geq 0$ is a fixed positive constant.

Pontryagin Maximum Principle: *if $(q(t), u(t))$ is solution of this optimal control problem then there is $\alpha(t) \in T^*_{q(t)}Q$, $q(t) \in Q$, s.t.*

$$\frac{d}{dt}\alpha(t) = X_{\widehat{H}_{u(t)}}(\alpha(t)), \qquad \widehat{H}(\alpha(t), u(t)) = \max_{u \in U} \widehat{H}(\alpha(t), u),$$

where $X_{\widehat{H}_u}$ Hamiltonian vector field defined by $\widehat{H}_u(\alpha) := \widehat{H}(\alpha, u)$.

If $p_0 \neq 0$, replacing $\alpha(t)$ by $\alpha(t)/p_0$ shows that in $\widehat{H}$ one can always assume that $p_0 = 1$. Solutions with $p_0 \neq 0$ are called **normal extremals**. Solutions with $p_0 = 0$ are called **abnormal extremals**. We work from now on only with normal extremals and set $p_0 = 1$.

Assume $\widehat{H} \in C^1$. Then the optimal control $u(t)$ is found by solving

$$\frac{\partial \widehat{H}}{\partial u}(\alpha(t), u(t)) = 0.$$

A sufficient condition that guarantees that maximum is achieved along the control $u(t)$ is that X is linear in u and g is strictly convex in u. In this case the optimal control is uniquely determined.

So, locally the Pontryagin Maximum Principle states that $\alpha(t) = (q(t), p(t))$ and $u(t)$ are determined by the system of equations

$$\frac{\partial \widehat{H}}{\partial u} = 0, \quad \dot{q} = \frac{\partial \widehat{H}}{\partial p} = X(q, u), \quad \dot{p} = -\frac{\partial \widehat{H}}{\partial q}.$$

If $\frac{\partial \widehat{H}}{\partial u} = 0$ can be solved for $u = u(\alpha)$, then these equations are the usual Hamilton equations for $H(\alpha) := \widehat{H}(\alpha, u(\alpha))$.

This happens locally, for example, if $\widehat{H}$ is of class C^2 and $\partial^2 \widehat{H}/\partial u^2 : U \to U^*$, computed at a given point, is an isomorphism. If X is linear in u and g is strictly convex in u, then this holds at every point.

These equations can be obtained by the variational principle

$$\delta \int_0^T \left(\widehat{H}(\alpha_q, u) - \langle \alpha_q, \dot{q} \rangle \right) dt = 0.$$

C.) Clebsch optimal control problem

$\Phi : G \times Q \to Q$ (left or right) action of a Lie group G on a manifold Q. **Infinitesimal generator** $u_Q \in \mathfrak{X}(Q)$ of the action for $u \in \mathfrak{g}$:

$$u_Q(q) := \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp(t\xi)}(q)$$

Clebsch optimal control problem: given $\ell : \mathfrak{g} \to \mathbb{R}$ find $u(t)$, $q(t)$

$$\min_{u(t)} \int_0^T \ell(u(t))dt,$$

ℓ **cost function**, subject to the following conditions:
(A) $\dot{q}(t) = u(t)_Q(q(t))$ or (A)' $\dot{q}(t) = -u(t)_Q(q(t))$;
(B) $q(0) = q_0$ and $q(T) = q_T$.

If (A)' is assumed instead of (A): *inverse representation*. Clebsch optimal control problem is obtained from standard one by choosing $U = \mathfrak{g}$, $g(q, u) = \ell(u)$, and $X(q, u) = u_Q(q)$. Thus, X is linear in u.

The Pontryagin function is in this case

$$\widehat{H}(\alpha_q, u) = \pm\langle \alpha_q, u_Q(q)\rangle - \ell(u) = \pm\langle \mathbf{J}(\alpha_q), u\rangle - \ell(u),$$

where the sign corresponds to (A) or (A)' and $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is the momentum map of the cotangent-lifted G-action on T^*Q.

Direct representation: *Assume that $u \in \mathfrak{g} \mapsto \frac{\delta \ell}{\delta u} \in \mathfrak{g}^*$ is a diffeomorphism. Let G act on the left (resp. right) on Q. Then, an extremal solution of the Clebsch optimal control problem with condition (A) is a solution of*

$$\frac{\delta \ell}{\delta u} = \mathbf{J}(\alpha), \quad \dot{\alpha} = u_{T^*Q}(\alpha).$$

Moreover, the solution reads $\alpha(t) = \Phi^{T^*}_{g(t)}(\alpha(0))$, where

$$\dot{g}(t)g(t)^{-1} = u(t), \quad \textit{resp.} \quad g(t)^{-1}\dot{g}(t) = u(t).$$

These equations imply Euler-Poincaré equations for the control u

$$\frac{d}{dt}\frac{\delta \ell}{\delta u} = -\operatorname{ad}^*_u \frac{\delta \ell}{\delta u}, \quad \textit{resp.} \quad \frac{d}{dt}\frac{\delta \ell}{\delta u} = \operatorname{ad}^*_u \frac{\delta \ell}{\delta u}.$$

Proof. For left action. $\frac{\partial \widehat{H}}{\partial u} = 0$ gives the condition $\frac{\delta \ell}{\delta u} = \mathbf{J}(\alpha)$ on the optimal control u, which can be solved to give $u = f(\alpha)$. Compute Hamilton's equations for $\widehat{H}_u : T^*Q \to \mathbb{R}$. Standard fact: the Hamiltonian vector field associated to the momentum function $\mathcal{P}(u)(\alpha_q) = \langle \alpha_q, u_Q(q) \rangle$ is

$$X_{\mathcal{P}(u)}(\alpha) = u_{T^*Q}(\alpha),$$

u_{T^*Q} infinitesimal generator of $\Phi_g^{T^*} := T^*\Phi_{g^{-1}}$ $(G \times T^*Q \to T^*Q)$.

Solution of $\dot{\alpha} = u_{T^*Q}(\alpha)$ is necessarily of the form $\alpha(t) = \Phi^{T^*}_{g(t)}(\alpha(0))$, where $g(0) = e$ and $\dot{g}(t)g(t)^{-1} = u(t)$. Therefore, since u is a function of α, we get $\dot{g}(t)g(t)^{-1} = f(\alpha(t)) = f\left(\Phi^{T^*}_{g(t)}(\alpha(0))\right)$ which is an ordinary differential equation for $g(t)$. We take the unique solution of this equation with initial condition $g(0) = e$. We thus obtain

$$\frac{\delta \ell}{\delta u(t)} = \mathbf{J}(\alpha(t)) = \mathbf{J}(\Phi^{T^*}_{g(t)}(\alpha(0))) = \mathrm{Ad}^*_{g(t)^{-1}} \mathbf{J}(\alpha(0)).$$

Differentiating with respect to t, get the Euler-Poincaré equations:

$$\frac{d}{dt}\frac{\delta \ell}{\delta u(t)} = -\,\mathrm{ad}^*_{\dot{g}(t)g(t)^{-1}} \frac{\delta \ell}{\delta u(t)} = -\,\mathrm{ad}^*_{u(t)} \frac{\delta \ell}{\delta u(t)}. \qquad \blacksquare$$

Inverse representation: *Assume* $u \in \mathfrak{g} \mapsto \frac{\delta \ell}{\delta u} \in \mathfrak{g}^*$ *diffeomorphism. Left (resp. right)* G *action on* Q. *Then, an extremal solution of Clebsch optimal control problem, condition (A)', is a solution of*

$$\frac{\delta \ell}{\delta u} = -\mathbf{J}(\alpha), \quad \dot{\alpha} = -u_{T^*Q}(\alpha).$$

Moreover, the solution reads $\alpha(t) = \Phi^{T^*}_{g(t)^{-1}}(\alpha(0))$, *where*

$$g(t)^{-1}\dot{g}(t) = u(t) \quad \textit{resp.} \quad \dot{g}(t)g(t)^{-1} = u(t).$$

These equations imply the Euler-Poincaré equations

$$\frac{d}{dt}\frac{\delta \ell}{\delta u} = \operatorname{ad}^*_u \frac{\delta \ell}{\delta u}, \quad \textit{resp.} \quad \frac{d}{dt}\frac{\delta \ell}{\delta u} = -\operatorname{ad}^*_u \frac{\delta \ell}{\delta u}.$$

Recall $H(\alpha) := \widehat{H}(\alpha, u(\alpha))$ where the optimal control $u(\alpha)$ is uniquely determined by the condition $\delta \ell / \delta u = \mathbf{J}(\alpha)$. We thus obtain

$$H(\alpha) = \left\langle \frac{\delta \ell}{\delta u}, u \right\rangle - \ell(u) = h\left(\frac{\delta \ell}{\delta u}\right) = h(\mathbf{J}(\alpha)),$$

where $h : \mathfrak{g}^* \to \mathbb{R}$ is the Hamiltonian associated to ℓ via the Legendre transformation $u \mapsto \delta \ell / \delta u$. So H is the **collective Hamiltonian** associated to h.

For (A)' (instead of (A)), the Hamiltonian is $H(\alpha) = h(-\mathbf{J}(\alpha))$.

Important example: $\ell(u) = \frac{1}{2}\|u\|^2$, where the norm is associated to a inner product on $\mathfrak{g}$. In this case, identifying the dual Lie algebra $\mathfrak{g}^*$ with $\mathfrak{g}$ via the inner product, we have

$$H(\alpha) = \frac{1}{2}\|\mathbf{J}(\alpha)\|^2.$$

QUESTION: What does this have to do with the convexity properties of the momentum map? Is the Sjamaar convexity theorem on cotangent bundles a particular case of very general convexity theorems, far beyond the symplectic category?

D.) Restriction to G-orbits, geodesics, and the normal metric

Simple observations:

- Canonical Hamilton equations $\dot{\alpha} = u_{T^*Q}(\alpha)$ on T^*Q induce canonical equations on $T^*\mathcal{O}$, where $\mathcal{O} := \{\Phi_g(q) \mid g \in G\}$ orbit.
- Theorems work for any G-manifold Q. So, in the Clebsch optimal control problem, can choose Q to be a G-orbit $\mathcal{O} \subset Q$.

• In particular, the solution of the Euler-Poincaré equations for ℓ are obtained by solving the canonical Hamilton's equations

$$\dot{\alpha} = u_{T^*\mathcal{O}}(\alpha)$$

on $T^*\mathcal{O}$. This was expected, since the solution is $\alpha(t) = T^*\Phi_{g(t)^{-1}}(\alpha_0)$ and thus preserves the G-orbits. Recall that the infinitesimal generator $u_{T^*\mathcal{O}} \in \mathfrak{X}(T^*\mathcal{O})$ is the Hamiltonian vector field associated to the canonical symplectic form $\Omega_{\mathcal{O}}$ on $T^*\mathcal{O}$ and to the momentum function $\mathcal{P}(u) \in \mathcal{F}(T^*\mathcal{O})$ defined by $\mathcal{P}(u)(\alpha_q) := \langle \alpha_q, \xi_{\mathcal{O}}(q) \rangle$.

Assume ℓ is the kinetic energy of a positive definite inner product γ on $\mathfrak{g}$. Given $q \in Q$, let $\mathfrak{g}_q := \{\xi \in \mathfrak{g} \mid \xi_Q(q) = 0\}$, isotropy Lie algebra at q. Orthogonal decomposition $\mathfrak{g} \ni \xi = \xi_q + \xi^q \in \mathfrak{g}_q \oplus \mathfrak{g}_q^\perp$. Get the normal Riemannian metric $\gamma_{\mathcal{O}}$ on the G-orbit through $q \in \mathcal{O}$

$$\gamma_{\mathcal{O}}(\xi_{\mathcal{O}}(q), \eta_{\mathcal{O}}(q)) := \gamma(\xi^q, \eta^q).$$

This formula recovers the usual normal metric on adjoint orbits on compact Lie algebras. If action is locally free: $\gamma_{\mathcal{O}}(\xi_{\mathcal{O}}(q), \eta_{\mathcal{O}}(q)) := \gamma(\xi, \eta)$, because $\mathfrak{g}_q = \{0\}$.

$H : T^*\mathcal{O} \to \mathbb{R}$ associated to $L : T\mathcal{O} \to \mathbb{R}$ given by $\gamma_{\mathcal{O}}$: Denoting by $\sharp$ the usual index raising operator associated to γ and $\gamma_{\mathcal{O}}$, we have

$$H(\alpha_q) = \frac{1}{2}\gamma_{\mathcal{O}}(\alpha_q^\sharp, \alpha_q^\sharp), \qquad \alpha_q \in T_q^*Q.$$

If the inner product γ is Ad-invariant, then the normal metric $\gamma_{\mathcal{O}}$ is G-invariant. This is a particular case of the earlier observation: an Ad-invariant Lagrangian ℓ induces G-invariant Hamiltonian H.

G acts on the left (resp. right) on Q, γ positive definite inner product on $\mathfrak{g}$, ℓ Lagrangian = kinetic energy. Then, an extremal solution of the Clebsch optimal control problem with condition (A) *is given by $u(t) = \mathbf{J}(\alpha(t))^\sharp$, where $\alpha(t)^\sharp$ projects to a geodesic on a G-orbit $\mathcal{O} \subset Q$, for the normal Riemannian metric $\gamma_{\mathcal{O}}$ on $\mathcal{O}$*

$$\gamma_{\mathcal{O}}(\xi_Q(q), \eta_Q(q)) := \gamma(\xi^q, \eta^q).$$

Moreover, this curve is given by $\alpha(t) = T^*\Phi_{g(t)^{-1}}(\alpha(0))$, where

$$\dot{g}(t)g(t)^{-1} = u(t), \quad \textit{resp.} \quad g(t)^{-1}\dot{g}(t) = u(t).$$

Remarkably, by Euler-Poincaré theory, $g(t)$ is a geodesic on G for the right (resp. left) invariant metric induced on G by γ.

E.) **The case of a Lie group: $Q = H$**

Specialize results if Q is a Lie group $H \supset G$ and the action is given by multiplication $G \times H \to H$. $H = G$ is permitted. Given $u \in \mathfrak{g}$, the infinitesimal generator associated to *left* multiplication by G on H is $u_H(q) = T_e R_q(u) =: uq$; for *right* multiplication: $u_H(q) = T_e L_q(u) =: qu$.

Clebsch optimal control problem: Given $q_0, q_T \in H$, find $u(t) \in \mathfrak{g}$, $q(t) \in \mathfrak{h}$ such that

$$\min_{u(t)} \int_0^T \ell(u(t))dt$$

subject to the following conditions:
(A) $\dot{q}(t) = u(t)q(t)$, resp. $\dot{q}(t) = q(t)u(t)$;
(B) $q(0) = q_0$ and $q(T) = q_T$

The associated variational principles are

$$\delta \int_0^T \left(\langle \alpha_q, uq - \dot{q} \rangle - \ell(u) \right) dt = 0, \quad \text{resp.} \quad \delta \int_0^T \left(\langle \alpha_q, qu - \dot{q} \rangle - \ell(u) \right) dt = 0,$$

the optimal control u is given by

$$\frac{\delta \ell}{\delta u} = \mathbf{J}(\alpha_q) = i^* \left(\alpha_q q^{-1} \right), \quad \text{resp.} \quad \frac{\delta \ell}{\delta u} = \mathbf{J}(\alpha_q) = i^* \left(q^{-1} \alpha_q \right),$$

$i^* : \mathfrak{h}^* \to \mathfrak{g}^*$ dual of $i : \mathfrak{g} \hookrightarrow \mathfrak{g}$ and Hamilton's equations on T^*H are

$$\dot{\alpha} = u\alpha, \quad \text{resp.} \quad \dot{\alpha} = \alpha u.$$

Solution: $\alpha(t) = g(t)\alpha_0$, resp. $\alpha(t) = \alpha_0 g(t)$. For the base curves: $q(t) = g(t)q_0$, resp. $q(t) = q_0 g(t)$. If $\alpha_0 \in T_e^* H$, then $q(t) = g(t)$.

What does the previous theorem state in this case? For left (resp. right) translation by G, the orbits are $\mathcal{O}_q = \{gq \mid g \in G\}$ (resp. $\mathcal{O}_q = \{qg \mid g \in G\}$), where $q \in H$ is fixed. Since the action is free, the normal metric is

$$\gamma_{\mathcal{O}_q}(uf, vf) = \gamma(u, v) \quad \text{resp.} \quad \gamma_{\mathcal{O}_q}(fu, fv) = \gamma(u, v), \quad f \in \mathcal{O}_q .$$

By the theorem, if the Lagrangian is given by the kinetic energy of γ, then the base curves $q(t) = g(t)q_0$, resp. $q(t) = q_0 g(t)$ are geodesics on $\mathcal{O}_{q_0}$ with respect to $\gamma_{\mathcal{O}_q}$. This is coherent with the Euler-Poincaré interpretation saying that $g(t)$ is a geodesic on G with respect to the G-invariant metric induced by γ.

If $h \in G$, then the orbit coincides with the subgroup, $\mathcal{O}_h = G$, and the normal metric is the right (resp. left) invariant extension of γ to G. In this case, the two interpretations of geodesics coincide.

In the inverse representation, condition (A) is replaced by (A)',

$$\dot{q} = -uq, \quad \text{resp.} \quad \dot{q} = -qu,$$

then the variational principles become

$$\delta \int_0^T \left(\langle \alpha_q, uq + \dot{q} \rangle + \ell(u)\right) dt = 0, \quad \text{resp.} \quad \delta \int_0^T \left(\langle \alpha_q, qu + \dot{q} \rangle + \ell(u)\right) dt = 0,$$

the optimal control u is given by

$$\frac{\delta \ell}{\delta u} = -\mathbf{J}(\alpha_q) = -i^*\left(\alpha_q q^{-1}\right), \quad \text{resp.} \quad \frac{\delta \ell}{\delta u} = -\mathbf{J}(\alpha_q) = -i^*\left(q^{-1}\alpha_q\right)$$

and the canonical Hamilton's equations are

$$\dot{\alpha} = -u\alpha, \quad \text{resp.} \quad \dot{\alpha} = -\alpha u.$$

However, in this case, if the initial condition α_0 lies at the identity, then the curves $q(t)$ and $g(t)$ of the theorem are related by

$$q(t) = g(t)^{-1}.$$

This explains why condition (A)' in the Clebsch optimal control problem is referred to as the "inverse representation". As we will see below, this point of view is important for fluids.

F.) The N dimensional free rigid body

Applying these results to $G = SO(N)$, $H = GL(N)$ recovers the *symmetric representation of the N-rigid body*. One can also consider $G = SO(N)$ acting on $Q = \mathfrak{gl}(N)$ by matrix multiplication on the right.

A vector in $TSO(N) \subset TGL(N) = GL(N) \times \mathfrak{gl}(N)$ is of the form (Q, V), where $Q \in SO(N)$, $V = QU$, and $U \in \mathfrak{so}(N)$. Identify the cotangent and tangent bundles via the pairing

$$\langle P, V \rangle := \mathrm{Tr}(P^T V),$$

which turns out to be a bi-invariant Riemannian metric on $SO(N)$ (in fact the extension of minus the Killing form on $\mathfrak{so}(N)$).

Optimal control problem: Given $Q_0, Q_T \in GL(N)$, find $U(t) \in \mathfrak{so}(N)$ and $Q(t) \in \mathfrak{gl}(N)$ such that

$$\min_{U(t)} \int_0^T \ell(U(t))dt$$

subject to the following conditions:
(A) $\dot{Q}(t) = Q(t)U(t)$
(B) $Q(0) = Q_0$ and $Q(T) = Q_T$.
This corresponds to the *right* action of $SO(N)$ on $GL(N)$, although the rigid body is *left* invariant; consistent with theorem, where *right* cotangent lifted actions produce the *left* Euler-Poincaré equations.

The cotangent bundle momentum map $\mathbf{J} : T^*GL(N) \to \mathfrak{so}(N)^* = \mathfrak{so}(N)$ (via pairing above) associated to right multiplication is

$$\mathbf{J}(Q,P) = \frac{1}{2}\left(Q^T P - P^T Q\right).$$

By Theorem, the optimal control U is found by solving the equation

$$\frac{\delta \ell}{\delta U} = \frac{1}{2}\left(Q^T P - P^T Q\right).$$

Note: if $(Q,P) \in T^*SO(N)$, then $\mathbf{J}(Q,P) = \frac{1}{2}\left(Q^T P - P^T Q\right) = Q^{-1}P$; recovers usual momentum map associated to reduction on the left. Lagrangian of the N-rigid body is $\ell(U) := \frac{1}{4}\langle U, JU\rangle$, where $J : \mathfrak{so}(N) \to \mathfrak{so}(N)$ symmetric positive definite operator. So,

$$\frac{\delta \ell}{\delta U} = \frac{1}{2} JU \quad \text{and the optimal control is} \quad U = J^{-1}\left(Q^T P - P^T Q\right).$$

Using the formula $(Q,P) \mapsto (Qg, Pg)$ for the cotangent lift of right translation by $SO(N)$ on $GL(N)$, we obtain the canonical Hamilton's equations on $T^*GL(N)$:

$$\dot{Q} = QU, \quad \dot{P} = PU.$$

Note that if $Q(0)$, $P(0) \in SO(N)$, then $P(t) \in SO(N)$ for all time, since $(Q(t), P(t)) = (Q(0)g(t), P(0)g(t))$, $g(t) \in SO(N)$. Thus $SO(N) \times SO(N)$ is an invariant submanifold. Recall from the general theory, that the Hamiltonian for these equations is $H(Q, P) = h\left(\frac{Q^T P - P^T Q}{2}\right)$. In the case of the rigid body, we get

$$H(P, Q) = \frac{1}{4}\left\langle \left(Q^T P - P^T Q\right), J^{-1}\left(Q^T P - P^T Q\right)\right\rangle .$$

These results coincide with those of Bloch-Brockett-Crouch (1996, 1997) and are obtained here as a particular case of the general Clebsch optimal control problem.

From general theory, we know that equations $\dot{Q} = QU, \dot{P} = PU$ are also Hamiltonian on the $SO(N)$ orbits in $GL(N)$, for any Lagrangian ℓ whose Legendre transform is a diffeomorphism. If ℓ is given by a kinetic energy, which is the case for the N-rigid body, then Q is a geodesic on an $SO(N)$ orbit relative to the normal metric. When the $SO(N)$-orbit in $GL(N)$ is precisely the subgroup $SO(N)$ of $GL(N)$, this geodesic interpretation coincides with the usual Euler-Poincaré approach.

G.) Optimal control for incompressible fluids

Apply results to the “Lie group" $H = \text{Diff}(\mathcal{D})$ of all diffeomorphisms of the compact Riemannian manifold $\mathcal{D}$ with boundary and its subgroup $G = \text{Diff}_{vol}(\mathcal{D})$ of volume preserving diffeomorphisms. We shall recover both the approaches of Bloch-Crouch-Holm-Marsden [2000] and Holm [2009] which appear now as particular cases of the two general theorems on the Clebsch Optimal Control Problem.

Recall that a curve $\eta_t \in \text{Diff}_{vol}(\mathcal{D})$ represents the Lagrangian motion of an ideal fluid in the domain $\mathcal{D}$, that is, the curve $\eta_t(x)$ in $\mathcal{D}$ is the trajectory of the fluid particle located at x at time $t = 0$, assuming that η_0 is the identity; η_t is referred to as the *forward map*.

The “Lie algebra" of G consists of divergence free vector fields on $\mathcal{D}$ tangent to the boundary and is denoted by $\mathfrak{g} = \mathfrak{X}_{vol}(\mathcal{D})$. The curve η_t is the flow of the Eulerian velocity $u_t \in \mathfrak{X}_{vol}(\mathcal{D})$, that is,

$$\dot{\eta}_t = u_t \circ \eta_t$$

The curve $l_t := \eta_t^{-1}$ is called the *back-to-label map* and is related to the Eulerian velocity u_t via the relation

$$\dot{l}_t + Tl_t \cdot u_t = 0.$$

Well known (Arnold [1966], Ebin-Marsden [1970]): *A curve $\eta_t \in$ Diff$_{vol}(\mathcal{D})$ is a geodesic with respect to the L^2 right invariant Riemannian metric if and only if u_t is a solution of the Euler equations*

$$\dot{u} + u \cdot \nabla u = - \operatorname{grad} p.$$

Thus, the Euler equations are the Euler-Poincaré equations on $\mathfrak{X}_{vol}(\mathcal{D})$ associated to the Lagrangian $\ell(u) = \frac{1}{2} \int_{\mathcal{D}} \|u\|^2$.

First approach: Use first general theorem, left version, to obtain the optimal control formulation of the Euler equations using the forward map η_t; this will give the result of Bloch-Crouch-Holm-Marsden [2000]. Given two diffeomorphisms $\eta_0, \eta_T \in$ Diff$(\mathcal{D})$, we consider the optimal control problem

$$\min_{u_t} \int_0^T \|u_t\|^2 dt$$

subject to the following conditions:

(A) $\dot{\eta}_t = u_t \circ \eta_t$;
(B) $\eta(0) = \eta_0$ and $\eta(T) = \eta_T$.

Use L^2 pairing induced by the Riemannian metric on $\mathcal{D}$, to identify the tangent and cotangent bundles of the diffeomorphism groups. Then, the momentum map $\mathbf{J} : T^* \operatorname{Diff}(\mathcal{D}) \to \mathfrak{X}_{vol}(\mathcal{D})^*$ associated to the cotangent-lift of left translation of $\operatorname{Diff}_{vol}(\mathcal{D})$ on $\operatorname{Diff}(\mathcal{D})$ is

$$\mathbf{J}(\eta, \pi) = \mathbb{P}(J\eta^{-1}(\pi \circ \eta^{-1})) = J\eta^{-1}(\pi \circ \eta^{-1}) - \operatorname{grad} k,$$

where $\mathbb{P} : \mathfrak{X}(\mathcal{D}) \to \mathfrak{X}_{div}(\mathcal{D})$ is the (L^2-orthogonal) Helmoltz-Hodge projector onto divergence free vector fields parallel to the boundary. The optimal control is thus given by $u = \pi \circ \eta^{-1} - \operatorname{grad} k$ and Hamilton's equations on $T^* \operatorname{Diff}(\mathcal{D})$ are

$$\dot{\eta} = u \circ \eta, \quad \dot{\pi} = -(Tu \circ \eta)^{\dagger} \pi,$$

where $\dagger$ means the transpose with respect to the Riemannian metric on $\mathcal{D}$. These equations can be obtained via the variational principle

$$\delta \int_0^T (\langle \pi, u \circ \eta - \dot{\eta} \rangle - \ell(u))\, dt = 0.$$

By general theory, u *verifies the Euler-Poincaré equations, producing here the ideal fluid motion* $\dot{u}+u\nabla u = -\operatorname{grad} p$. Recovers optimal control formulation of Euler equations given by Bloch et al [2000].

In a similar way as for the N-rigid body, *Hamilton's equations are equivalent to the geodesics spray of the normal metric on the tangent bundle of a* $\text{Diff}_{vol}(\mathcal{D})$*-orbit. Solution is the cotangent-lift acting on the initial condition* π_0*, that is,* $\pi = (T\eta^{-1})^{\dagger} \circ \pi_0$*.*

Second approach: Apply second general theorem, right version, to obtain the optimal control formulation for Euler fluid equations, via the back-to-label map $l_t = \eta_t^{-1}$; this will give the result of Holm [2009]. Given two diffeomorphisms $l_0, l_T \in \text{Diff}(\mathcal{D})$, we consider the optimal control problem

$$\min_{u_t} \int_0^T \|u_t\|^2 dt$$

subject to the following conditions:

(A)] $\dot{l}_t + Tl_t \circ u_t = 0$
(B) $l(0) = l_0$ and $l(T) = l_T$.

The momentum map $\mathbf{J} : T^* \operatorname{Diff}(\mathcal{D}) \to \mathfrak{X}_{vol}(\mathcal{D})^*$ for the cotangent-lift of right translation is

$$\mathbf{J}(l,\pi) = \mathbb{P}(Tl^\dagger \circ \pi) = Tl^\dagger \circ \pi - \operatorname{grad} q.$$

The optimal control is thus given by $u = -Tl^\dagger \circ \pi + \operatorname{grad} q$ and the Hamilton's equations on $T^* \operatorname{Diff}(\mathcal{D})$ are

$$\dot{l} = -Tl \circ u, \quad \dot{\pi} = -T\pi \circ u.$$

These equations can be obtained via the variational principle

$$\delta \int_0^T \Big(\langle \pi, Tl \circ u + \dot{l} \rangle + \ell(u) \Big)\, dt = 0.$$

By general theory, u verifies the Euler-Poincaré equations yielding ideal fluid motion $\dot{u} + u \cdot \nabla u = -\operatorname{grad} p$. This recovers the optimal control formulation of Euler equations given by Holm [2009].

Comparison

N**-dimensional free rigid body:**

$$\dot{Q} = QU, \quad \dot{P} = PU, \quad JU = \frac{1}{2}(Q^TP - P^TQ) = \mathbb{P}(Q^TP)$$

Ideal incompressible Euler flow:

$$\dot{l} = -Tl \circ u, \quad \dot{\pi} = -T\pi \circ u, \quad u = -\mathbb{P}(Tl^\dagger \circ \pi)$$

$\mathbb{P} : \mathfrak{gl}(N) \to \mathfrak{so}(N)$, resp. the Helmoltz projector $\mathbb{P} : \mathfrak{X}(\mathcal{D}) \to \mathfrak{X}_{vol}(\mathcal{D})$.

For fluids, can also replace group $H = \text{Diff}(\mathcal{D})$ by the manifold $Q = \text{Emb}(\mathcal{D}, M)$ of all embeddings of M into a fixed manifold M. In fact, the Euler fluid equations, resp. the N-rigid body equations, can be obtained by a Clebsch optimal control problem on any manifold Q on which the group $G = \text{Diff}_{vol}(\mathcal{D})$, resp. $G = SO(N)$ act. Interpretation in terms of geodesics on $\text{Diff}_{vol}(\mathcal{D})$-orbits holds as before.

H.) Averaged hydrodynamics

Replacing the L^2 metric by an H^1 metric on $\text{Diff}_{vol}(\mathcal{D})$, one gets the dynamics of averaged Euler (or Euler-α) equations given by

$$\dot{m} + u\cdot\nabla m - \alpha^2 \nabla u^T \cdot \Delta u = -\,\text{grad}\, p, \quad m = (1-\alpha^2\Delta)u.$$

Both the approaches described before for ideal fluids are directly applicable to averaged dynamics. It suffices to use the Lagrangian $\ell(u) = \frac{1}{2}\|u\|_{H^1}$ associated to the Sobolev H^1 norm. The optimal controls are respectively given by

$$u = (1-\alpha^2\Delta)^{-1}(\pi \circ \eta^{-1}) \quad \text{and} \quad u = -(1-\alpha^2\Delta)^{-1}(Tl^\dagger \circ \pi).$$

I.) Optimal control for the N-Camassa-Holm equation, singular solutions

The N-Camassa-Holm equations

$$\dot{m} + u\cdot\nabla m + \nabla u^T \cdot m + m\, \text{div}\, u = 0, \quad m = (1-\alpha^2\Delta)u$$

are the spatial representation of geodesics on the group $\text{Diff}(\mathcal{D})$ of all diffeomorphisms of $\mathcal{D}$, relative to the Sobolev H^1 metric.

So, they are obtained by Euler-Poincaré reduction; particular case of the EPDiff equations. As for Euler equations, obtain equations by two different Clebsch optimal control problems. Remarkably, the first approach recovers the dynamics of singular solutions and gives a new interpretation of these singular solutions as geodesics.

First approach: Use first general theorem, left version, with $G = \mathrm{Diff}(\mathcal{D})$ acting on the left on the manifold $Q = \mathrm{Emb}(S, \mathcal{D})$ of all embeddings of a given manifold S into M. Given two embeddings $\mathbf{Q}_0, \mathbf{Q}_T \in \mathrm{Emb}(S, \mathcal{D})$, associated Clebsch optimal control problem is

$$\min_{u_t} \int_0^T \|u_t\|^2_{H^1} dt \tag{1}$$

subject to the following conditions:
(A) $\dot{\mathbf{Q}}_t = u_t \circ \mathbf{Q}_t$;
(B) $\mathbf{Q}(0) = \mathbf{Q}_0$ and $\mathbf{Q}(T) = \mathbf{Q}_T$.

The momentum map $\mathbf{J} : T^* \mathrm{Emb}(S, \mathcal{D}) \to \mathfrak{X}(\mathcal{D})^*$ is given by

$$\mathbf{J}(\mathbf{Q}, \mathbf{P}) = \int_S \mathbf{P}(s) \delta(x - \mathbf{Q}(s)) ds;$$

Since $\mathbf{J}$ produces singular solutions, the generalizations of the peakons of the one dimensional Camassa-Holm equation, $\mathbf{J}$ is often called the **singular momentum map**. The optimal control is hence

$$(1-\alpha^2\Delta)u = \int_S \mathbf{P}(s)\delta(x-\mathbf{Q}(s))ds,$$

and Hamilton's equations are obtained from the variational principle

$$\delta \int_0^T (\langle \mathbf{P}, u \circ \mathbf{Q} - \dot{\mathbf{Q}}\rangle - \ell(u))dt = 0.$$

Evaluated on the optimal control u, the Pontryagin Hamiltonian $\widehat{H}$ produces the collective Hamiltonian

$$H(\mathbf{Q},\mathbf{P}) = \frac{1}{2}\iint \mathbf{P}(s)G(\mathbf{Q}(s)-\mathbf{Q}(s'))\mathbf{P}(s')dsds'$$

where G is the Green's function associated to the differential operator $(1-\alpha^2\Delta)$. By general theory, the solution $(\mathbf{Q},\mathbf{P})$ is obtained by letting the flow η_t of the optimal control act on the initial values $\mathbf{Q}(0),\mathbf{P}(0)$ by the cotangent-lifted action. The fact that optimal control u is solution of the N-Camassa-Holm equations, recovers the interpretation of the momentum map as a singular solution.

Apply the first general theorem to this case: new geometric interpretation of singular solutions.

The singular solutions $\delta\ell/\delta u = \mathbf{J}(\mathbf{Q}, \mathbf{P})$ of the N-Camassa-Holm equations arise as normal metric geodesics on a $\operatorname{Diff}(\mathcal{D})$*-orbit*

$$\mathcal{O} = \{\eta \circ \mathbf{Q}_0 \mid \eta \in \operatorname{Diff}(\mathcal{D})\} \subset \operatorname{Emb}(S, \mathcal{D}),$$

$$\gamma_{\mathcal{O}}(u \circ \mathbf{Q}, v \circ \mathbf{Q}) := \langle u, v \rangle_{H^1}, \quad \forall \mathbf{Q} \in \mathcal{O}.$$

Note that choosing $S = \mathcal{D}$, we have $\operatorname{Emb}(S, \mathcal{D}) = \operatorname{Diff}(\mathcal{D})$ and we recover the dynamic of the strong (i.e. non singular) solutions.

Second approach: Apply second general theorem, right version, in order to obtain the optimal control formulation for Euler fluid equations, via a generalization of the back-to-label map.

$G = \operatorname{Diff}(\mathcal{D})$ acts on the right on $\operatorname{Emb}(\mathcal{D}, M)$ for a fixed manifold M. Given $\mathbf{q}_0, \mathbf{q}_T \in \operatorname{Emb}(\mathcal{D}, M)$, Clebsch optimal control problem is

$$\min_{u_t} \int_0^T \|u_t\|^2 dt$$

subject to the following conditions:

(A) $\dot{\mathbf{q}}_t + T\mathbf{q} \circ u_t = 0$;

(B) $\mathbf{q}(0) = \mathbf{q}_0$ and $\mathbf{q}(T) = \mathbf{q}_T$.

The momentum map $\mathbf{J} : T^* \mathrm{Emb}(\mathcal{D}, M) \to \mathfrak{X}(\mathcal{D})^*$ is given by

$$\mathbf{J}(\mathbf{q}, \mathbf{p}) = \mathbf{p} \cdot \mathbf{dq}.$$

Thus, the optimal control is given by

$$(1 - \alpha^2 \Delta) u = \mathbf{p} \cdot \mathbf{dq},$$

Hamilton's equations are obtained via the variational principle

$$\delta \int_0^T (\langle \mathbf{p}, \mathbf{dq} \circ u + \dot{\mathbf{q}} \rangle + \ell(u))\, dt = 0,$$

and the collective Hamiltonian reads

$$H(\mathbf{q}, \mathbf{p}) = h(\mathbf{p} \cdot \mathbf{dq}) = \frac{1}{2} \iint \mathbf{p}(x) \cdot \mathbf{dq}(x) G(x - x') \mathbf{p}(x') \cdot \mathbf{dq}(x') dx dx'.$$

In this case, the second general theorem yields the following interpretation of the solution $(\mathbf{q}, \mathbf{p})$.

The solution $(\mathbf{q}, \mathbf{p})$ of Hamilton's equations defined by H projects to a geodesic on a Diff$(\mathcal{D})$*-orbit*

$$\mathcal{O} = \{\mathbf{q}_0 \circ \eta \mid \eta \in \mathrm{Diff}(\mathcal{D})\},$$

with respect to the normal metric

$$\gamma_{\mathcal{O}}(T\mathbf{q} \circ u, T\mathbf{q} \circ v) = \langle u, v \rangle_{H^1}, \quad \forall \mathbf{q} \in \mathcal{O}.$$

J.) The case of the adjoint action

G acts on $Q := \mathfrak{g}$ on the right : $x \mapsto \mathrm{Ad}_{g^{-1}} x$. The infinitesimal generator is $u_{\mathfrak{g}}(x) = [x, u]$. Consider the Lagrangian $\ell : \mathfrak{g} \to \mathbb{R}$, defined by $\ell(u) := \frac{1}{2}\|u\|^2$, where the norm is taken relative to an Ad-invariant nondegenerate symmetric bilinear form γ on $\mathfrak{g}$, so

$$\gamma([u, x], v) = \gamma(u, [x, v]).$$

Medina-Roy [1985] give complete classification of such Lie algebras. Assume, in addition, that γ is positive definite inner product.

Given $x_0, x_T \in \mathfrak{g}$, associated Clebsch optimal control problem is

$$\min_{u(t)} \int_0^T \frac{1}{2}\|u(t)\|^2 dt$$

subject to the following conditions:
(A) $\dot{x} = [x, u]$;
(B) $x(0) = x_0$ and $x(T) = x_T$.

Use the first general theorem: $Q = \mathfrak{g}$, G acts on on the right on $\mathfrak{g}$ by $u \mapsto \mathrm{Ad}_{g^{-1}} u$. Identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ via the the inner product γ, the Pontryagin function is $\widehat{H}(x, p, u) = \langle p, [x, u] \rangle - \frac{1}{2}\|u\|^2$ and the momentum map $\mathbf{J} : T^*\mathfrak{g} \to \mathfrak{g}^*$ is $\mathbf{J}(x, p) = -[x, p]$. Thus the optimal control is $u = [p, x]$ and the canonical Hamilton's equation are

$$\dot{x} = [x, u], \quad \dot{p} = [p, u],$$

thus, we get the double bracket equations

$$\dot{x} = [x, [p, x]], \quad \dot{p} = [p, [p, x]].$$

By general theory, these equations are Hamiltonian on $T^*\mathfrak{g}$, for

$$H(x,p) = \langle p, [x,[p,x]]\rangle - \frac{1}{2}\|[p,x]\|^2 = \frac{1}{2}\|[p,x]\|^2.$$

By the first general theorem, the control u necessarily satisfies the Euler-Poincaré equations which in this case are $\dot{u} = -\operatorname{ad}_u u = 0$ since with the identification by the bi-invariant inner product γ we have $\operatorname{ad}_u^* = -\operatorname{ad}_u$. Hence the control u is constant along the flow of Hamilton's equations, that is, $u = [p,x]$ is a constant of the motion.

By the normal metric theorem, the double bracket equations are also the Hamiltonian description of geodesics on an adjoint orbit $\mathcal{O} = \{\operatorname{Ad}_g \xi \mid g \in G\}$ relative to the normal metric $\gamma_{\mathcal{O}}$. In this case of the adjoint action, it has the expression

$$\gamma_{\mathcal{O}}([x,p],[x,q]) = \gamma(p^x, q^x),$$

where the decomposition $p = p_x + p^x$ is made relative to the splitting $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{g}_x^{\perp}$. If $\mathfrak{g}$ is a compact semisimple Lie algebra and γ is minus the Killing form, this recovers the usual normal metric on adjoint orbits.

Note that here we have

$$\mathfrak{g}_x = \ker(\mathrm{ad}_x) \quad \text{and} \quad (\mathfrak{g}_x)^{\perp} = \mathrm{im}(\mathrm{ad}_x)\,.$$

Note that, by our general theory, a similar result holds for any Lagrangian ℓ such that $u \mapsto \delta\ell/\delta u$ is a diffeomorphism. More precisely, Hamilton's equations for $H(x,p) = h([p,x])$ on $T^*\mathfrak{g}$ restrict to Hamilton's equation on $T^*\mathcal{O}$.

As for the rigid body, one can consider the kinetic energy associated to $\gamma(u, Jv)$, where $J : \mathfrak{g} \to \mathfrak{g}$ is a symmetric and positive definite operator. In this case, the optimal control is given by $u = J^{-1}[p,x]$, the Hamiltonian is $H(x,p) = \frac{1}{2}\gamma([x,p], J^{-1}[x,p])$, and Hamilton's equations read

$$\dot{x} = [x, J^{-1}[p,x]], \quad \dot{p} = [p, J^{-1}[p,x]].$$

The inner product $\gamma(u, Jv)$ induces a normal metric on orbits in the same way as before. Geodesics of this metric are given by the above Hamilton's equations.

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